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ON DIOPHANTINE PROBLEMS INVOLVING FRACTIONAL POWERS OF INTEGERS

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Abstract

In this thesis we are concerned with Diophantine problems of fractional degree. First we consider diagonal Diophantine inequalities of the shape

$$\left|\lambda_1 x_1^\theta + \cdots + \lambda_s x_s^\theta\right| < \tau,$$

where $\theta > 2$ is real and non-integral, λ_i are non-zero real numbers not all of the same sign and τ is a positive real number. For such inequalities we obtain an asymptotic formula for the number of positive integer solutions $\mathbf{x} = (x_1, \ldots, x_s)$ inside a box of side length P. Moreover, we consider the problem of representing a large positive real number by a positive definite generalised polynomial of degree θ. Our approach follows the Davenport–Heilbronn–Freeman method. A key element in our proof is an essentially optimal mean value estimate for an exponential sum involving fractional powers of integers.

We then turn our attention to systems of simultaneous equations and inequalities. Let λ_i, μ_j be non-zero real numbers not all of the same sign and let a_i, b_k be non-zero integers not all of the same sign. We investigate a mixed Diophantine system of the shape

$$\begin{cases} \left|\lambda_1 x_1^\theta + \cdots + \lambda_\ell x_\ell^\theta + \mu_1 y_1^\theta + \cdots + \mu_m y_m^\theta\right| < \tau \\ a_1 x_1^d + \cdots a_\ell x_\ell^d + b_1 z_1^d + \cdots + b_n z_n^d = 0, \end{cases}$$

where $d \geq 2$ is an integer, $\theta > d+1$ is real and non-integral and τ is a positive real number. For such systems we obtain an asymptotic formula for the number of positive integer solutions $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x_1, \ldots, z_n)$ inside a bounded box. Our approach makes use of a two-dimensional version of the classical Hardy–Littlewood circle method and the Davenport–Heilbronn–Freeman method. The proof involves a combination of essentially optimal mean value estimates for the auxiliary exponential sums, together with estimates stemming from the classical Weyl and Weyl–van der Corput inequalities.

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Chapter 1

Introduction

1.1 General introduction to the thesis

As the title suggests, this thesis is concerned with the study of Diophantine problems involving fractional powers of integers. Ignoring for now the adjective *fractional*, let us first explain the term *Diophantine problems*. That is, problems concerning the investigation of existence of integer solutions to equations with integer coefficients. The name comes from the ancient Greek mathematician Diophantus of Alexandria who is the author of *Arithmetica*, the oldest surviving book on the topic of algebraic equations.

Suppose that s is a natural number. A specific type of Diophantine equations that has attract a lot of attention is equations of the diagonal shape

$$c_1x_1^k + \cdots + c_sx_s^k = 0, \tag{1.1.1}$$

where c_i are non-zero integers and $k \geq 2$ is a natural number. If one were to replace the integers c_i by some non-zero real numbers λ_i then it makes sense to ask whether the inequality

$$|\lambda_1x_1^k + \cdots + \lambda_sx_s^k| < \epsilon, \tag{1.1.2}$$

possesses a non-trivial solution $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{Z}^s$ for arbitrarily small values of $\epsilon > 0$. The term non-trivial refers to solutions $\mathbf{x}$ with at least one component $x_i \neq 0$. Note that due to homogeneity the existence of a solution implies that the inequality actually possesses infinitely many integer solutions. The main theme of the present thesis is the study of inequalities of the shape (1.1.2), where now we consider a non-integral exponent θ, namely we study inequalities of the shape

$$|\lambda_1x_1^\theta + \cdots + \lambda_sx_s^\theta| < \epsilon, \tag{1.1.3}$$

where $\theta > 2$ is real and non-integral. As in the case of equations, one may consider studying systems of inequalities. Here we investigate a mixed system consisting of an equation and an inequality. More specifically, we investigate the simultaneous solubility of equations of the shape (1.1.1) and inequalities of the shape (1.1.3).

Before we close this first section we introduce some pieces of notation that we use in the rest of this work. For $x \in \mathbb{R}$ we write $e(x)$ to denote $e^{2\pi i x}$ with $i = \sqrt{-1}$ being the imaginary unit. For a complex number z we write $\overline{z}$ to denote its complex conjugate. For a function $f : \mathbb{Z} \to \mathbb{C}$ and for two real numbers m, M, whenever we write

$$\sum_{m<x\leq M} f(x)$$

the summation is to be understood over the integers that belong to the interval $(m, M]$.

We make use of the standard symbols of Vinogradov and Landau. Namely, when for two functions f, g there exists a positive real constant C such that $|f(x)| \leq C|g(x)|$ for all sufficiently large x we write $f(x) = O(g(x))$ or $f(x) \ll g(x)$. We write $f \asymp g$ to denote the relation $g \ll f \ll g$. Furthermore, we write $f(x) = o(g(x))$ if $f(x)/g(x) \to 0$ as $x \to \infty$ and we write $f \sim g$ if $f(x)/g(x) \to 1$ as $x \to \infty$. For a real number x we shall write $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$ and $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ to denote the ceiling and the floor function respectively. For two integers a, b we write (a, b) to denote their greatest common divisor. An expression of the shape $m < \mathbf{x} \leq M$ where $m < M$ and $\mathbf{x} = (x_1, \dots, x_n)$ is an n-tuple, is to be understood as $m < x_1, \dots, x_n \leq M$. Similarly, for n tuples $\mathbf{x}, \mathbf{y}, \mathbf{z}$, an inequality of the shape $\mathbf{y} < \mathbf{x} \leq \mathbf{z}$ is to be understood as $y_i < x_i \leq z_i$ for $1 \leq i \leq n$. Finally, and for the sake of clarity, let us declare at this stage that the term *form* refers to a homogeneous polynomial. Moreover, we say that a form is *non-degenerate* if all of its coefficients are non-zero. Recall as well that the term *non-trivial solution* refers to a solution $\mathbf{x} \neq \mathbf{0}$.

1.2 Diagonal Diophantine inequalities

One of the first major results in the study of additive type Diophantine inequalities is due to Davenport and Heilbronn [30]. In this paper it is proven that any real indefinite diagonal quadratic form in 5 variables that is not proportional to a form with integral coefficients, can take arbitrarily small values. Let us make this more concrete.

Theorem 1.2.1 (Davenport and Heilbronn – [30]). *Suppose that $\lambda_1, \dots, \lambda_5$ are non-zero real numbers, not all of the same sign, and such that one at least of the ratios λ_i/λ_j is irrational. We write*

$$Q(\mathbf{x}) = \lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2.$$

Then there exist arbitrarily large integers P such that the inequality

$$|Q(\mathbf{x})| < 1$$

has more than γP^3 solutions with $1 \leq x_1, \dots, x_5 \leq P$. Here $\gamma = \gamma(\lambda_1, \dots, \lambda_5)$ is a positive real constant.

Note here that one may obtain a corresponding result for the inequality $|Q(\mathbf{x})| < \epsilon$ for any given $\epsilon > 0$ by simple applying the theorem to the quadratic form $\epsilon^{-1}Q(\mathbf{x})$. In order to prove Theorem 1.2.1, Davenport and Heilbronn developed a variant of the classical Hardy–Littlewood circle method. This method, which is now called the Davenport–Heilbronn method, has been since then a fundamental tool in studying the solubility of inequalities.

Let us now say a word behind the motivation in the investigation of Davenport and Heilbronn. In 1884 Meyer [47] proved that any non-degenerate indefinite quadratic form of the shape

$$q(\mathbf{x}) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 \in \mathbb{Z}[\mathbf{x}]$$

possesses a non-trivial rational (and hence by clearing the denominators an integral) zero **x**. Note that this is sharp. Consider the quadratic form

$$q(\mathbf{x}) = x_1^2 + x_2^2 - 3(x_3^2 + x_4^2).$$

Using Fermat's method of infinite descent and reducing $\pmod 3$ one can readily see that the equation $q(\mathbf{x}) = 0$ does not admit a non-trivial integer solution. Motivated by Meyer's result, Oppenheim [52] in 1929 conjectured that any real non-degenerate indefinite quadratic form Q in $s \geq 5$ variables which is not proportional to a rational form can take arbitrarily small values. In 1953, in a series of papers [53], [54], [55], Oppenheim made the stronger conjecture that for a quadratic form as above in $s \geq 3$ variables, the set $Q(\mathbb{Z}^s)$ is dense in $\mathbb{R}$. Oppenheim's conjecture, now a theorem after Margulis's work [46], is usually formulated as follows.

Theorem 1.2.2 (Oppenheim Conjecture–Margulis's Theorem [46])**.** *Let Q be a non-degenerate real indefinite quadratic form in $s \geq 3$ variables, that is not proportional to a form with rational coefficients. Then for any $\epsilon > 0$ there exists $\mathbf{x} \in \mathbb{Z} \setminus \{\mathbf{0}\}$ such that $0 < |Q(\mathbf{x})| < \epsilon$.*

The case of real indefinite diagonal quadratic forms was investigated by Davenport and Heilbronn in [30]. In 1956 Davenport [27] proved that any indefinite real quadratic form can take arbitrarily small values, provided that it can be expressed as a sum of squares of real linear forms that have sufficiently many positive and negative signs. This result was subsequently improved by Davenport [28] in 1958, and by Birch and Davenport [7]. Further improvements were obtained by Davenport and Ridout [33] and Ridout [66]. To give an idea of the spirit of these results, let us describe Ridout's conclusion. Suppose that Q is a real indefinite quadratic form in s variables. Suppose further that after a diagonalization Q can be expressed a sum of squares of real linear forms with r positive signs and $s - r$ negative signs. If $\min(r, s - r) \leq 4$ and $s \geq 21$, then for any $\epsilon > 0$ the inequality $|Q(\mathbf{x})| < \epsilon$ has a non-trivial integer solution.

Oppenheim's conjecture was finally settled in the affirmative in 1989 by Margulis [46]. Margulis's approach in proving Oppenheim's conjecture makes use of different tools and methods than the previous mentioned works.

As a closing remark on our short discussion about Oppenheim's conjecture, let us say that Margulis's result is the best possible. There exists real indefinite quadratic forms in 2 variables that have no non-trivial integer solutions. Set

$$Q(x_1, x_2) = x_1^2 - \alpha^2 x_2^2,$$

where $\alpha = 1 + \sqrt{2}$. Note that the quadratic form we consider is diagonal. Using Liouville's theorem [1] we can show that there exists a constant $C > 0$ so that for any pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$

[1]see [44, Theorem 191, §11.7] ; Suppose that α is an irrational that is a root of a polynomial $f \in \mathbb{Z}[x]$ of degree $d > 0$. Then there exists a real constant $C > 0$ such that for all pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ one has $|\alpha - p/q| > C/q^d$.

one has

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^2}.$$

By changing signs if necessary we may assume that $\frac{x_1}{x_2} > 0$. So by the above inequality one has

$$\frac{C}{x_2^2}\alpha < \left|\alpha - \frac{x_1}{x_2}\right| \cdot \left|\alpha + \frac{x_1}{x_2}\right|,$$

which in turn yields that the inequality $|Q(x_1, x_2)| < \epsilon$ does not admit a non-trivial integer solution for $\epsilon \leq C\alpha$.

We return now to the setting of diagonal inequalities. As we already discussed, Davenport and Heilbronn [30] proved that if λ_i are non-zero real numbers not all of the same sign, and such that at least one of the ratios λ_i/λ_j is irrational, then the inequality

$$\left|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2\right| < \epsilon$$

possesses a non-trivial solution in positive integers for any $\epsilon > 0$. Moreover, in [30] it is pointed out that using Hua's inequality one may similarly prove (under the same assumptions on λ_i) that if $k \geq 2$ is a fixed integer, then the inequality

$$|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k| < \epsilon \tag{1.2.1}$$

possesses a non-trivial solution in positive integers for any $\epsilon > 0$, provided that $s \geq 2^k + 1$. For the details of the proof, the interested reader may look at [78, Chapter 11]. In §1.8 we shall give a sketch of this argument.

Following the work of Davenport and Heilbronn, various results were obtained concerning diagonal inequalities. In 1955 Davenport and Roth [34] used Vinogradov's estimate for Weyl sums to show that for $k \geq 12$ the inequality (1.2.1) is non-trivially soluble in positive integers provided that $s \geq Ck \log k$, where C is an absolute positive constant. Moreover, in the same paper, it is proven that if M is an arbitrary real number, then for any $\epsilon > 0$ the inequality

$$\left|\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3 + M\right| < \epsilon$$

has infinitely many solutions in positive integers. In fact, Brüdern [14] showed that if there is at least one irrational ratio λ_i/λ_j, then the inequality

$$\left|\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3\right| < (\max|x_i|)^{\epsilon - 1/4}$$

has infinitely many integer solutions. Baker, Brüdern and Wooley [3] showed that any diagonal cubic inequality in $s = 7$ variables that is not proportional to an integral form has infinitely many integer solutions. More precisely, in [3] it is proven if there is at least one irrational ratio λ_i/λ_j, then for any $M \in \mathbb{R}$ the inequality

$$\left|\lambda_1 x_1^3 + \cdots + \lambda_7 x_7^3 - M\right| < (\max|x_i|)^{-10^{-4}}$$

has infinitely many solutions in integers. Under the extra condition that there is a ratio λ_i/λ_j which is irrational and algebraic, Brüdern [17] replaced the exponent -10^{-4} by -360^{-1}. Moreover, for cubic inequalities Brüdern in [16] investigated the size of the solutions in terms of the coefficients of the form. The main result of [16] says that if $\lambda_i \geq 1$ are real numbers, then for any $\epsilon > 0$ there exists $\mathbf{x} \in \mathbb{Z}^8$ such that the inequalities

$$\left|\lambda_1 x_1^3 + \cdots + \lambda_8 x_8^3\right| < 1$$

and

$$0 < \sum_{i=1}^{8} \lambda_i |x_i|^3 \ll (\lambda_1 \cdots \lambda_8)^{15/8+\epsilon}$$

hold simultaneously. This improved previous work of Pitman and Ridout [63] on bounding solutions of cubic equations and inequalities.

Though in the present thesis we do not deal with mixed power inequalities, let us mention here that some results for such a case have been obtained by Brüdern in [13] and [15]. Suppose that $s, k_1, \ldots, k_s \geq 2$ are fixed natural numbers. The main theme of these papers is to show that for specific values of s, k_i and for integral x_i, the values taken by non-degenerate real forms of the shape

$$H(\mathbf{x}) = \lambda_1 x_1^{k_1} + \cdots + \lambda_s x_s^{k_s}$$

are dense in $\mathbb{R}$.

We shall close this section we a few comments on general cubic inequalities. For cubic not necessarily diagonal inequalities we have the work of Pitman [61]. Suppose that $C(\mathbf{x})$ is real cubic form in s variables. Pitman proved in [61] that for any $\epsilon > 0$ the inequality $|C(\mathbf{x})| < \epsilon$ possesses a non-trivial integer solution provided that $s \geq (1314)^{256} - 1$. This was the first finite lower bound for the number of variables needed to ensure the existence of a non-trivial integral zero for a real cubic form. Pitman's result was significantly improved by Freeman in [37] who showed that any real cubic form is non-trivially soluble in integers, provided that the number of variables s satisfies $s \geq 359, 551, 882$. For general real forms of odd degree, it was proven by Schmidt [67] that given enough variables it is always possible to prove the existence of a non-trivial integer solution. However, Schmidt gives no explicit value for the number of variables needed to ensure solubility. We shall come back to Schmidt's result in §1.5.

1.3 Asymptotic lower bounds and asymptotic formulas

Suppose that $s, k \geq 2$ are fixed integers. Suppose further that λ_i are fixed non-zero real numbers, such that at least one of the ratios λ_i/λ_j is irrational, and such that, if k is even, then not all have the same sign. We put

$$F(\mathbf{x}) = \lambda_1 x_1^k + \cdots + \lambda_s x_s^k.$$

Let P be a large positive real number and let $\tau > 0$ be a fixed real number. We denote by $N(P)$ the number of integer solutions of the inequality

$$|F(\mathbf{x})| < \tau, \tag{1.3.1}$$

with $|\mathbf{x}| \leq P$, where $|\mathbf{x}| = \max_i |x_i|$. There are about P^s possible choices for a tuple $\mathbf{x}$ with $|\mathbf{x}| \leq P$. Moreover, and roughly speaking, an arbitrary tuple $\mathbf{x}$ is a solution of the inequality (1.3.1) with probability P^{-k}. So, heuristically one expects that the number of solutions counted by $N(P)$ is roughly of the order of magnitude P^{s-k}. As we mentioned in the opening of §1.2, when $k=2$ and $s=5$ Davenport and Heilbronn in [30] showed that there exist arbitrarily large non-trivial integer solutions $\mathbf{x}$ to the inequality (1.3.1). Theorem 1.2.1 guarantees the existence of a sequence of integers $(P_n)_{n\in\mathbb{N}}$ with $P_n \to \infty$ as $n \to \infty$, such that $N(P_n) \gg \gamma P_n^{s-2}$ for some positive real number γ depending at most on λ_i and τ. *This asymptotic lower bound is proven to be valid for arbitrarily large values of P and not for all large values of P.* In the proof P is restricted to take values from the convergent continued fractions approximation of some irrational ratio λ_i/λ_j.

It becomes apparent from the above discussion that a central problem now is to establish an asymptotic lower bound for the counting function $N(P)$, that is of the correct order of magnitude for all sufficiently large P. This problem was open up until 2000. At this point Freeman [36] finally succeed to remove the previous restriction on the values of P. In [36] was established for the first time an asymptotic lower bound of the shape $N(P) \gg P^{s-k} \quad (P \to \infty)$. We state Freeman's result below.

Theorem 1.3.1 (Freeman – [36]). *Suppose that $k \geq 3$ is a fixed integer. Let*

$$s_0(k) = \min\left\{2^k+1, k(\log k + \log\log k + 3) + \frac{\widetilde{C}k\log\log k}{\log k}\right\},$$

where $\widetilde{C} > 0$ is a suitable absolute real constant. Suppose that $s \geq s_0(k)$ is an integer. Then as $P \to \infty$ one has

$$N(P) \gg P^{s-k},$$

where the implicit constant depends at most on k, s, τ and the coefficients λ_i.

A key role in the proof of Theorem 1.3.1 is played by the work of Bentkus and Götze [4] on value distribution of positive definite quadratic forms. Drawing inspiration from some of the methods of [4], Freeman developed further the Davenport–Heilbronn method in order to deliver a lower bound for the counting function $N(P)$ for all sufficiently large values of P. The apparatus introduced in [36] is now known as Freeman's variant of the Davenport–Heilbronn method. We illustrate the main ideas of this method in §1.8. An asymptotic formula for the counting function was obtained by Freeman in [39]. Building on [36] and introducing appropriate kernel functions, Freeman proved that given $s \geq 2^k+1$ variables one has as $P \to \infty$ that

$$N(P) = C(s,k;\boldsymbol{\lambda})\tau P^{s-k} + o\left(P^{s-k}\right), \tag{1.3.2}$$

where $C(s,k;\boldsymbol{\lambda})$ is a positive real number, which depends at most on k, s and λ_i.

For the inhomogeneous case, Freeman [40] extended the results of [36] in order to deal with additive inequalities of the shape

$$|h(x_1) + \cdots + h(x_s) - M| < \epsilon,$$

where h_i are real polynomials in one variable, of degree at most k. Using a diminishing ranges argument, Freeman proved that the above inequality has infinitely many integer solutions for any given real numbers ϵ, M with $\epsilon > 0$, provided that $s \geq s_0(k)$ where $s_0(k) \sim 4k \log k$. Note that compared to the homogeneous case, here for large k one needs four times more variables. This is due to the presence of the real number M which prevents one from making use of the technology for exponential sums over smooth numbers as in [36].

Back to the homogeneous case now, Freeman's results from [36] and [39] were improved shortly afterwards by Wooley [86]. We write $F(k)$ to denote the least integer s_0 so that whenever $s \geq s_0$ the asymptotic lower bound

$$N(P) \gg C(s,k;\boldsymbol{\lambda})\tau P^{s-k}$$

holds for all large P. As before, $C(s,k;\boldsymbol{\lambda})$ is a positive real number, which depends at most on k, s and λ_i. In [86] Wooley refined Freeman's approach by using an amplification procedure. In [86, Theorem 2] it is proven that when k is large one has

$$F(k) \leq k(\log k + \log\log k + 2 + o(1)),$$

while for small values of k one has $F(k) \leq \mathfrak{F}(k)$, where the integer $\mathfrak{F}(k)$ is recorded in the following tables. As far as the author is aware, this is the current state of art in the existing bibliography for the number $F(k)$.

Table 1.1: Values of $\mathfrak{F}(k)$ for $3 \leq k \leq 11$

k	3	4	5	6	7	8	9	10	11
$\mathfrak{F}(k)$	7	12	18	25	33	42	50	59	67

Table 1.2: Values of $\mathfrak{F}(k)$ for $12 \leq k \leq 20$

k	12	13	14	15	16	17	18	19	20
$\mathfrak{F}(k)$	76	84	92	100	109	117	125	134	142

Let us notice that a possible direction for future research here would be to improve on the values of $\mathfrak{F}(k)$, recorded on the above tables. It seems possible to obtain some improvement by using methods of [93].

In the same work [86, Corollary] Wooley showed that the asymptotic formula (1.3.2) is valid whenever

$$s \geq 2^k \ (k \geq 3), \qquad s \geq \frac{7}{8}2^k \ (k \geq 6), \qquad s \geq k^2(\log k + \log\log k + O(1)) \quad \text{when } k \text{ is large.}$$

Here one can get an improvement by using the latest developments in Vinogradov's mean value theorem due to Wooley [95]. For $\alpha \in \mathbb{R}$ we put

$$f_k(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k).$$

In [95, Corollary 14.7] it is proven that

$$\int_0^1 |f_k(\alpha)|^s \,\mathrm{d}\alpha \ll P^{s-d},$$

provided that $s \geq s_0$, where

$$s_0 = k^2 - k + 2\lfloor\sqrt{2k+2}\rfloor - \theta(k),$$

with $\theta(k)$ defined via

$$\theta(k) = \begin{cases} 1, & \text{when } \; 2k+2 \geq \lfloor\sqrt{2k+2}\rfloor^2 + \lfloor\sqrt{2k+2}\rfloor, \\ 2, & \text{when } \; 2k+2 < \lfloor\sqrt{2k+2}\rfloor^2 + \lfloor\sqrt{2k+2}\rfloor. \end{cases}$$

Combining the above estimate with the methods of [21], one may establish (1.3.2) using s_0+1 variables. This improves the previous results of Wooley for $s \geq 5$.

1.4 Diophantine inequalities of fractional degree

We may now come to describe the first result of this thesis. Instead of a diagonal form of degree $k \geq 2$ we consider a diagonal generalised polynomial of *fractional degree*. More specifically, suppose that $\theta > 2$ is real and non-integral, and suppose that s is a positive integer. Let $\lambda_1, \dots, \lambda_s$ be fixed non-zero real numbers not all of the same sign. Consider the generalised polynomial

$$\mathcal{F}(\mathbf{x}) = \lambda_1 x_1^\theta + \dots + \lambda_s x_s^\theta. \tag{1.4.1}$$

Note that in contrast to the previous case, we do not need to assume the existence of an irrational ratio λ_i/λ_j.

Going back to the literature, it seems that the first to consider studying additive problems with non-integral exponents is Segal in the early 1930's. In [69], [70] and [71] (see also [72]), Segal studied Waring's problem with non-integral exponents. Suppose that ν is a positive real number. Segal considered the inequality

$$|x_1^\theta + \dots + x_s^\theta - \nu| < \epsilon,$$

with $\theta > 2$ real and non-integral and $0 < \epsilon < \nu^{-c(\theta)/\theta}$, where $0 < c(\theta) < 1$ is a fixed number depending only on θ. For large values of ν Segal showed the existence of a solution $\mathbf{x} \in \mathbb{N}^s$, provided that we are given $s \geq s_0(\theta)$ variables, where $s_0(\theta) \approx \theta(\lfloor\theta\rfloor + 1)2^{\lfloor\theta\rfloor+1} + 1$.

Let $\tau > 0$ be a fixed real number. Recall from (1.4.1) the definition of the generalised polynomial $\mathcal{F}(\mathbf{x})$. We write $\mathcal{N}^\tau_{s,\theta}(P)$ to denote the number of integer solutions of the inequality

$$|\mathcal{F}(\mathbf{x})| < \tau, \tag{1.4.2}$$

with $1 \leq x_1, \dots, x_s \leq P$. In Chapter 2 we establish an asymptotic formula for the counting function $\mathcal{N}^\tau_{s,\theta}(P)$ as $P \to \infty$. Our result reads as follows.

Theorem 1.4.1. *Suppose that $\theta > 2$ is real and non-integral, and suppose further that $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) + 1$ is a natural number. Then as $P \to \infty$ one has*

$$\mathcal{N}_{s,\theta}^{\tau}(P) = 2\tau\Omega(s,\theta;\boldsymbol{\lambda})P^{s-\theta} + o\left(P^{s-\theta}\right),$$

where

$$\Omega(s,\theta;\boldsymbol{\lambda}) = \left(\frac{1}{\theta}\right)^{s} |\lambda_1 \cdots \lambda_s|^{-1/\theta} C(s,\theta;\boldsymbol{\lambda}) > 0$$

with

$$C(s,\theta;\boldsymbol{\lambda}) = \int_{\mathcal{U}} \left(-\sigma_s(\sigma_1\beta_1 + \cdots + \sigma_{s-1}\beta_{s-1})\right)^{1/\theta - 1} \left(\beta_1 \cdots \beta_{s-1}\right)^{1/\theta-1} \mathrm{d}\boldsymbol{\beta},$$

where $\mathrm{d}\boldsymbol{\beta}$ here stands for $\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_{s-1}$, and $\sigma_i = \lambda_i/|\lambda_i|$, and $\mathcal{U}$ denotes the set of points of the box $[0,|\lambda_1|] \times \cdots \times [0,|\lambda_{s-1}|]$, satisfying the condition that

$$-\sigma_s(\sigma_1\beta_1 + \cdots + \sigma_{s-1}\beta_{s-1}) \in [0,|\lambda_s|].$$

In particular, the inequality (1.4.2) possesses a non-trivial positive integer solution.

With minor adjustments the method we employ in proving Theorem 1.4.1 allows us to treat also positive definite generalised polynomials of the shape (1.4.1). A well known and extensively studied problem in additive number theory is Waring's problem. Suppose that $k \geq 2$ is a fixed integer. In its most standard form, the problem asks for the least natural number $s = s(k)$ such that every sufficiently large natural N is represented in the shape

$$N = x_1^k + \cdots + x_s^k,$$

where x_i are non-negative integers. A real analogue of this problem was studied by Chow in [23] and [24]. Using the Davenport–Heilbronn–Freeman method, Chow studied the number of solutions $\mathbf{x} \in \mathbb{N}^s$ as $\tau \to \infty$ of the inequality

$$\left|(x_1 - \theta_1)^k + \cdots + (x_s - \theta_s)^k - \tau\right| < \eta,$$

where $\theta_i \in (0,1)$ with $\theta_1 \notin \mathbb{Q}$ and $\eta \in (0,1]$ being fixed. In Chapter 2 we obtain a similar in spirit result.

Suppose that $\lambda_i > 0$ $(1 \leq i \leq s)$. For a positive real number ν sufficiently large in terms of s, k and τ, we ask how many positive integer solutions are possessed by the inequality

$$|\mathcal{F}(\mathbf{x}) - \nu| < \tau, \tag{1.4.3}$$

with $\mathcal{F}$ as in (1.4.1). We write $\rho_s(\tau,\nu) = \rho_s(\tau,\nu;\boldsymbol{\lambda})$ to denote the number of positive integer solutions of (1.4.3). One anticipates $\rho_s(\tau,\nu)$ to be large when τ is fixed and ν is large in terms of s, k, λ_i and τ. In Chapter 2 we prove the following result.

Theorem 1.4.2. *Suppose that $\theta > 2$ is real and non-integral, and that $\tau \in (0,1]$ is a fixed real number. Suppose further that $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) + 1$ is a natural number. Then as $\nu \to \infty$ one has*

$$\rho_s(\tau,\nu) = 2(\lambda_1 \cdots \lambda_s)^{-1/\theta} \frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)} \tau\nu^{s/\theta-1} + o\left(\nu^{s/\theta-1}\right).$$

Though the asymptotic formulae of Theorem 1.4.1 and Theorem 1.4.2 look similar, there is an essential difference between them. In the indefinite case of Theorem 1.4.1 we consider boxes of arbitrarily large side length P and so the main term grows like $P^{s-\theta}$. On the other hand, in the definite case covered by Theorem 1.4.2 , the main term in the asymptotic formula is growing like $\nu^{s/\theta-1}$ and hence it is limited by the size of the real number ν we wish to represent. This is explained by the fact that there is a natural height restriction imposed on a solution $\mathbf{x}$ of the inequality (1.4.3).

1.5 Systems of Diophantine inequalities

In this section we shall be concerned with systems of inequalities and mixed systems consisting of both equations and inequalities. For this reason let us begin our discussion with systems of integral forms. For diagonal integral forms we have the work of Davenport and Lewis [32]. In this paper the authors study the zeros of simultaneous forms of the shape

$$G_i(\mathbf{x}) = a_{i1}x_1^k + \cdots + a_{is}x_s^k \qquad (1 \leq i \leq R),$$

where a_{ij} are fixed integers and $k \geq 2$ is a fixed integer. In [32] it was proven that simultaneously the forms $G_i(\mathbf{x})$ $(1 \leq i \leq R)$ possess a non-trivial integer zero provided that the number of variables s satisfies $s \geq \lfloor 9R^2 k \log(3Rk) \rfloor$ when k is odd and $s \geq \lfloor 48R^2k^3 \log(3Rk^2) \rfloor$ when k is even. This paper has been influential for many subsequent works dealing with simultaneous zeros of forms and inequalities.

One of the most remarkable results concerning systems of general integral forms is due to Birch [6]. Suppose that $F_1, \ldots, F_R$ are any integral forms of degree k in s variables. Then the system of equations $F_i(\mathbf{x}) = 0$ $(1 \leq i \leq R)$ possesses a non-trivial integer solution provided that we have enough variables. Here enough variables means that s exceeds a quantity that grows at least quadratically with R. This was recently improved by Myerson in [49] and [50], to a linear dependence on R in the case of quadratic and cubic forms.

Suppose now that instead of integral forms we consider real forms. More relevant to us is the case of simultaneous diagonal inequalities. However, before we discuss this case and present some of the existing results in the literature, we choose to say a word for systems of general (not necessarily diagonal) real forms of unlike degree. This is certainly much harder compared to the diagonal situation of like degrees. A major result here dating to 1980 is due to Schmidt [67], who investigated the solubility of simultaneous general real forms of unlike odd degrees. Below we quote one version of Schmidt's result. Let us remark that the following version is not the strongest conclusion that was established in [67]. However, it is good enough for us to illustrate the spirit of the results obtained in [67].

Theorem 1.5.1 (Schmidt – [67]). *Let $h \geq 1$ be a given integer and let E be a given positive real number. Suppose that $d_1, \ldots, d_h$ are given odd integers and suppose further that $F_i(\mathbf{x})$ is a real form of degree d_i, for $1 \leq i \leq h$. Then there exists a positive real number $\tau = \tau(d_1, \ldots, d_h, E)$ such that for any natural number $s \geq \tau$ and any real number $N \geq 1$, there exists $\mathbf{x} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ which satisfies*

$$\max_{1 \leq i \leq s} |x_i| \leq N,$$

and

$$|F_i(\mathbf{x})| \ll N^{-E}|F_i| \qquad (1 \le i \le h),$$

where $|F_i|$ stands for maximum of the absolute values of the coefficients of the form F_i. The implicit constant depends only on $d_1, \dots, d_h$ and E.

Roughly speaking, Schmidt's result tells us that any system of real forms of odd degree has a non-trivial integer solution provided that we are given enough variables. However, no explicit value was given for the number $\tau = \tau(d_1, \dots, d_h, E)$. For diagonal real forms of like odd degree we have the following result due to Nadesalingam and Pitman.

Theorem 1.5.2 (Nadesalingam and Pitman – [51])**.** *Suppose that $R \ge 2$ and $d \ge 13$ are integers with d being odd. Suppose further that $F_1, \dots, F_R$ are diagonal real forms in $s \ge \lceil R^2 k^2 \log(3Rk) \rceil$ variables. Then for any $\epsilon > 0$ the system of inequalities*

$$|F_i(\mathbf{x})| < \epsilon \qquad (1 \le i \le R)$$

has a non-trivial integer solution.

The result of Nadesalingam and Pitman contains implicitly the case where the forms are multiplies of rational forms. In such a case and for sufficiently small ϵ, some of the inequalities are reduced to equations with integer coefficients. The authors combine the classical circle method of Hardy and Littlewood with the Davenport–Heilbronn method. A key step in their approach is the use of an inductive argument on the number of integral forms hidden in the system.

Suppose now that we are given any R real forms $F_i(\mathbf{x})$ $(1 \le i \le R)$ in s variables of degree d, where d is an odd integer. We write $\tau(d, R)$ to denote the least positive natural number s_0 such that for $s \ge s_0$ and any $\epsilon > 0$ the system of simultaneous inequalities $|F_i(\mathbf{x})| < \epsilon$ $(1 \le i \le R)$ possesses a non-trivial integer solution. The first finite upper bound for the number $\tau(3, R)$ was given by Freeman in the early 2000's. In [42] Freeman proved that $\tau(3, h) \le (10h)^{\gamma}$ where $\gamma = (10h)^5$. Freeman's approach goes through the pursuit of bounded non-trivial integer solutions to cubic inequalities. The proof makes use of diagonalization techniques and builds on [51].

We turn now our attention to systems consisting of even degree forms. We begin with the simplest case, namely a pair of two quadratic forms. We write

$$Q_i(\mathbf{x}) = \lambda_{i1}x_1^2 + \cdots + \lambda_{is}x_s^2 \qquad (i = 1, 2),$$

where λ_{ij} are fixed real numbers. In 1974 Cook [26] considered the system of simultaneous quadratic inequalities

$$|Q_i(\mathbf{x})| < \epsilon \qquad (i = 1, 2)$$

in $s = 9$ variables. In order to exclude the case where the forms Q_1 and Q_2 are multiplies of rational forms, Cook associated to the forms Q_1 and Q_2 the following ternary linear forms. For $1 \le i < j < k \le 9$ we write

$$L_{ijk}(u, v, w) = \det \begin{pmatrix} u & v & w \\ \lambda_{1i} & \lambda_{1j} & \lambda_{1k} \\ \lambda_{2i} & \lambda_{2j} & \lambda_{2k} \end{pmatrix}.$$

With this piece of notation we may now state the result of Cook.

Theorem 1.5.3 (Cook – [26]). *Suppose that Q_1 and Q_2 have real algebraic coefficients in nine variables. Suppose further that*

(i) Every member of the pencil $\{\alpha Q_1 + \beta Q_2\}$ with $(\alpha, \beta) \in \mathbb{R} \setminus \{\mathbf{0}\}$ is an indefinite form with at least five non-zero coefficients ;

(ii) Not all the ternary linear forms $L_{ijk}(u, v, w)$ have coefficients which are linearly dependent over the rationals.

Then for every $\epsilon > 0$ there exist integers $x_1, \ldots, x_9$ not all zero, such that

$$|Q_1(\mathbf{x})| < \epsilon \qquad \text{and} \qquad |Q_2(\mathbf{x})| < \epsilon.$$

In [18] Brüdern and Cook considered a pair of real diagonal cubic forms in $s = 15$ variables. Suppose that F_1 and F_2 are two such forms. Assuming that the coefficients of F_1 and F_2 are algebraic and the condition (ii) of Theorem 1.5.3 is satisfied, they showed that for any $\epsilon > 0$ the system of inequalities $|F_i(\mathbf{x})| < \epsilon$ $(i = 1, 2)$ has infinitely many integer solutions. This improved a result of Pitman [61]. Moreover, in [19] Brüdern and Cook considered the solubility of R simultaneous real diagonal forms of odd degree $k \geq 2$. Say $F_1, \ldots, F_R$ are such forms. Imposing a condition similar to the condition (ii) of Theorem 1.5.3, they proved that the system of inequalities $|F_i(\mathbf{x})| < \epsilon$ $(1 \leq i \leq R)$ has a non-trivial integer solution provided that the number of variables is of order Rn_0 with $n_0 = O(k \log k)$. As a matter of fact, Brüdern and Cook proved an asymptotic lower bound of the correct magnitude for the number of solutions inside a box of side length P. However, this asymptotic lower bound is proven to be valid for a sequence of arbitrarily large positive real numbers P and not for all large P. This is due to the limitation of the Davenport–Heilbronn method as it was introduced in [30].

Returning to the case of quadratic forms, the next result comes from Freeman. Using his variant variant of the Davenport–Heilbronn method, Freeman considered in [36] a system of R diagonal real quadratic forms. Below we state that result for the case $R = 2$.

Theorem 1.5.4 (Freeman – [38]). *Let Q_1, Q_2 be real diagonal quadratic forms in $s \geq 10$ variables. Suppose that every member of the pencil $\{\alpha Q_1 + \beta Q_2\}$ with $(\alpha, \beta) \in \mathbb{R} \setminus \{\mathbf{0}\}$ has at least five non-zero coefficients, one irrational coefficient, at least one negative coefficient and at least one positive coefficient. Then for any $\epsilon > 0$ there exists $\mathbf{x} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ such that*

$$|Q_1(\mathbf{x})| < \epsilon \qquad \text{and} \qquad |Q_2(\mathbf{x})| < \epsilon. \tag{1.5.1}$$

Moreover, if P is a sufficiently large positive real number and $N(P)$ denotes the number of integer solutions of the system (1.5.1) with $1 \leq \mathbf{x} \leq P$, then as $P \to \infty$ one has

$$N(P) \gg P^{s-2},$$

where the implicit constant depends on ϵ and the coefficients of the forms Q_1 and Q_2.

One may compare the result of Freeman with that of Cook. The latter requires fewer variables to ensure solubility. However, this comes with stronger assumptions on the coefficients of the forms. The assumption that the forms Q_1, Q_2 have algebraic coefficients and the condition (ii) in Theorem 1.5.3, are now being replaced by the condition that for any $(\alpha, \beta) \in \mathbb{R} \setminus \{ \mathbf{0} \}$ the form $\alpha Q_1 + \beta Q_2$ has at least one irrational coefficient. In [38, §2] it is demonstrated that Cook's condition is in fact strictly stronger than the above irrationality condition. For the case of $R > 2$ simultaneous real diagonal quadratic forms, Freeman [38, Theorem 3] obtained a similar result in $s \geq 5R$ variables. For this conclusion Freeman made the following assumptions : the existence of a non-singular real solution, a rank condition on the coefficients of the forms and an irrationality condition of the shape

$$\begin{aligned}&\text{for every choice } (\beta_1, \ldots, \beta_R) \in \mathbb{R}^R \setminus \{\mathbf{0}\}, \text{ the form } \beta_1 Q_1 + \cdots + \beta_R Q_R \\ &\text{has at least one irrational coefficient.}\end{aligned} \tag{1.5.2}$$

The next step in Freeman's investigations is to remove the irrationality condition (1.5.2). Note that this condition ensures that no form in the pencil is an integral form. Hence, subject to such a condition, one considers only true inequalities and excludes systems consisting of equations and inequalities. Suppose that $R \geq 2$ is a fixed integer. We put

$$F_i(\mathbf{x}) = \lambda_{i1} x_1^k + \cdots + \lambda_{is} x_s^k \qquad (1 \leq i \leq R),$$

where λ_{ij} are fixed real numbers and $k \geq 2$ is a fixed integer. We may reduce the system consisting of the forms F_i into a new system of forms (which by abuse of notation we denote again by F_i), for which the following condition is satisfied. Suppose that for some $0 \leq r \leq R$ the forms $F_1, \ldots, F_r$ are integral and suppose further that if $(\beta_1, \ldots, \beta_R) \in \mathbb{R}^R$ is such that the form $\beta_1 F_1 + \cdots + \beta_R F_R$ is rational then $\beta_{r+1} = \cdots = \beta_R$. Furthermore, in the case where k is even suppose that the system of equations $F_1(\mathbf{x}) = \cdots = F_R(\mathbf{x}) = 0$ possesses a non-singular real solution. Subject to a suitable rank condition and if the singular series associated to the integral forms $F_1, \ldots, F_r$ is positive, it was proven by Freeman [41] that the system of inequalities

$$|F_i(\mathbf{x})| < \epsilon \qquad (1 \leq i \leq R), \tag{1.5.3}$$

has a non-trivial integer solution for any $\epsilon > 0$, provided that the number of variables s satisfies $s \geq Rn_0$, where n_0 is of order of magnitude $k \log k + o(k \log k)$. At the cost of extra variables, Freeman demonstrates that the assumption for the singular series to be positive may be dropped. The proof combines techniques by Nadesalingam and Pitman [51], together with methods of Bentkus and Götze [4], as in [36]. If we write $N(P)$ to denote the number of solutions of the system (1.5.3) with $|\mathbf{x}| \leq P$, Freeman shows that

$$N(P) \gg P^{s-Rk} \qquad (P \to \infty),$$

where the implicit constant depends on s, k, R, ϵ and the coefficients of the forms F_i.

We now come to discuss the case of systems of inequalities of unlike degrees. The first to consider the simultaneous solubility of real forms of unlike degree was Parsell in [56]. In that paper, motivated by Wooley's work on simultaneous additive equations [83], [85], and using Wooley's methods on exponential sums over smooth numbers [84], Parsell developed a two dimensional version of the Davenport–Heilbronn method to study the solubility of a system of pair of diagonal cubic and quadratic form. Shortly afterwards, and taking advantage of Freeman's work [36], Parsell in [57] obtained an asymptotic lower bound for the number of solutions of a pair of cubic and quadratic form lying inside a box of sufficiently large side length. In the third paper on that series [59], Parsell considered a more general system consisting of R diagonal forms of unlike degree. Fix an integer $R \geq 2$. Let $k_1 \geq \cdots \geq k_R \geq 2$ be fixed integers and let λ_{ij} be fixed non-zero real numbers. Suppose that $\tau > 0$ is a fixed real number. The subject of [59] is the solubility in integers of the simultaneous inequalities

$$\left|\lambda_{1i}x_1^{k_i} + \cdots + \lambda_{is}x_s^{k_i}\right| < \tau \qquad (1 \leq i \leq s). \tag{1.5.4}$$

No irrationality assumption is being made here. Hence, there might be a case that some of the forms are actually proportional to an integral form. Parsell calls this the integral subsystem and makes the assumption that it possesses a p-adic solution for every prime p. Moreover, Parsell makes the assumption that the system

$$\lambda_{1i}x_1^{k_i} + \cdots + \lambda_{is}x_s^{k_i} = 0 \qquad (1 \leq i \leq s)$$

possesses a non-trivial real solution. For technical reasons associated with the application of the circle method these local solutions are assumed to be non-singular. For a system which satisfies this local solubility condition define $\widehat{G}^\star(\mathbf{k})$ to be the least number number s such that whenever $s \geq s_0$ the system (1.5.4) has a non-trivial integer solution. The main theorem of [59] examines the relationship between mean value estimates for exponential sums and the bounds these estimates yield for the number $\widehat{G}^\star(\mathbf{k})$. As a corollary, in the case $R = 2$ Parsell obtained some explicit values for small exponents k_1 and k_2. Below we record the values obtained in [59, Corollary 1.2].

Table 1.3: Values of $\widehat{G}^\star(k_1, k_2)$ for $k_1 \in \{3, 4, 5\}$ and $k_2 \in \{2, 3, 4\}$

(k_1, k_2)	$(3, 2)$	$(4, 2)$	$(4, 3)$	$(5, 2)$	$(5, 3)$	$(5, 4)$
$\widehat{G}^\star(k_1, k_2)$	13	20	24	31	32	36

Table 1.4: Values of $\widehat{G}^\star(k_1, k_2)$ for $k_1 \in \{6, 7\}$ and $k_2 \in \{3, 4, 5, 6\}$

(k_1, k_2)	$(6, 3)$	$(6, 4)$	$(6, 5)$	$(7, 4)$	$(7, 5)$	$(7, 6)$
$\widehat{G}^\star(k_1, k_2)$	49	47	50	65	64	66

For $(k_1, k_2) = (3, 2)$ using methods of [91] we expect to be able to get $\widehat{G}^\star(3, 2) \leq 11$. Further improvements might be possible using some of the methods of [12].

1.6 A mixed system of an inequality and an equation

We now come to describe our result for a mixed system consisting of an equation and an inequality. We fix non-zero real numbers λ_i, μ_j not all of the same sign and non-zero integers a_i, b_k not all of the same sign. Suppose that $d \geq 2$ is an integer and suppose further that $\theta > d+1$

is real and non-integral. We write

$$\begin{cases} \mathfrak{F}(\mathbf{x},\mathbf{y}) = \lambda_1 x_1^\theta + \cdots + \lambda_\ell x_\ell^\theta + \mu_1 y_1^\theta + \cdots + \mu_m y_m^\theta \\ \\ \mathfrak{D}(\mathbf{x},\mathbf{z}) = a_1 x_1^d + \cdots a_\ell x_\ell^d + b_1 z_1^d + \cdots + b_n z_n^d. \end{cases} \tag{1.6.1}$$

Let $\tau > 0$ be a fixed real number. The system we study is of the shape

$$\begin{cases} |\mathfrak{F}(\mathbf{x},\mathbf{y})| < \tau \\ \\ \mathfrak{D}(\mathbf{x},\mathbf{z}) = 0. \end{cases} \tag{1.6.2}$$

Here we make the following local solubility assumptions as in [59]. We ask for the system

$$\mathfrak{F}(\mathbf{x},\mathbf{y}) = \mathfrak{D}(\mathbf{x},\mathbf{z}) = 0 \tag{1.6.3}$$

to admit a non-trivial real solution $(\mathbf{x},\mathbf{y},\mathbf{z})$. Moreover, we ask $\mathfrak{D}(\mathbf{x},\mathbf{z}) \equiv 0 \pmod{p^\nu}$ to be soluble for all prime powers p^ν. Additionally, we impose the extra condition that the local solutions are in fact non-singular. When all these assumptions hold, then we say that the system (1.6.2) satisfies the local solubility condition.

We write $\mathcal{N}(P)$ to denote the number of positive integer solutions $(\mathbf{x},\mathbf{y},\mathbf{z})$ of the system (1.6.2) with

$$\frac{1}{2}\mathbf{x}^\star P < \mathbf{x} \le 2\mathbf{x}^\star P, \quad \frac{1}{2}\mathbf{y}^\star P < \mathbf{y} \le 2\mathbf{y}^\star P, \quad \frac{1}{2}\mathbf{z}^\star P < \mathbf{z} \le 2\mathbf{z}^\star P,$$

where $(\mathbf{x}^\star,\mathbf{y}^\star,\mathbf{z}^\star)$ is a non-singular real solution of the system of equations (1.6.3). In Chapter 3 we obtain an asymptotic formula for the counting function $\mathcal{N}(P)$ as $P \to \infty$. Our result reads as follows.

Theorem 1.6.1. *Suppose that $d \ge 2$ is an integer and suppose further that $\theta > d+1$ is real and non-integral. Let τ be a fixed positive real number. Consider the system*

$$|\mathfrak{F}(\mathbf{x},\mathbf{y})| < \tau \quad \textit{and} \quad \mathfrak{D}(\mathbf{x},\mathbf{z}) = 0, \tag{1.6.4}$$

with $\mathfrak{F},\mathfrak{D}$ defined in (1.6.1). We write

$$A_\theta = (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) \quad \textit{and} \quad A_d = d^2.$$

Moreover, we set

$$s_{\min} = \left\lceil \max\left\{ A_\theta + n,\ \frac{A_d}{A_\theta} m + A_\theta \right\} \right\rceil + 1$$

and

$$s_{\max} = \left\lfloor \min\left\{ A_\theta + A_d,\ A_\theta + \frac{A_d}{A_\theta} m + n \right\} \right\rfloor + 1.$$

Suppose that the system (1.6.4) satisfies the following conditions.

(a) The system (1.6.4) satisfies the local solubility condition, namely the system (1.6.3) possesses a non-singular real solution and the congruence $\mathfrak{D}(\mathbf{x},\mathbf{z}) \equiv 0 \pmod{p^\nu}$ possesses a non-

singular solution for all prime powers p^ν.

(b) *One has $\ell \geq \max\{\lceil 2\theta(1-n/d)\rceil,\ 1\},\ 0 \leq m \leq A_\theta$ and $0 \leq n \leq A_d$.*

(c) *One has $\ell + m \geq A_\theta + 1$ and $\ell + n \geq A_d + 1$.*

(d) *For the total number of variables $s = \ell + m + n$ one has $s_{\min} \leq s \leq s_{\max}$.*

Then, there exists a positive real number $C = C(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \theta, d, s)$, such that as $P \to \infty$ one has

$$\mathcal{N}(P) = 2\tau C P^{s-(\theta+d)} + o\left(P^{s-(\theta+d)}\right).$$

In particular, the number of positive integer solutions $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in [1,P]^\ell \times [1,P]^m \times [1,P]^n$ of the system (1.6.4) is $\gg P^{s-(\theta+d)}$, where the implicit constant is a positive real number, which depends on $s, \lambda_i, \mu_j, a_i, b_k, \theta, d$ and τ.

Note here that by the assumptions made in Theorem 1.6.1 our conclusions are valid when the total number of variables $s = \ell + m + n$ satisfies $A_\theta + 1 \leq s \leq A_\theta + A_d + 1$. The method we employ in Chapter 3 allows one to handle systems with $s \geq A_\theta + A_d + 2$ variables and show that in such a case the number of positive integer solutions of the system (1.6.4) inside a box of side length P is $\gg P^{s-(\theta+d)}$ as $P \to \infty$.

1.7 Vinogradov's mean value theorem

We now say a few words about the methods we employ in proving our results. Until the mid 1930's the standard approach for estimating exponential sums was by means of repeated differencing process, as introduced by Weyl [82] and van der Corput [75]. In 1935, Vinogradov [79] (see also [80]) came up with a new method. Let $s, k \in \mathbb{N}$. For $\boldsymbol{\alpha} \in \mathbb{R}^k$ we write

$$f_k(\boldsymbol{\alpha}; P) = \sum_{1 \leq x \leq P} e(\alpha_1 x + \cdots + \alpha_k x^k).$$

In order to study the exponential sum $f_k(\boldsymbol{\alpha}; P)$, Vinogradov turned his attention into the mean value

$$J_{s,k}(P) = \int_{[0,1)^k} |f_k(\boldsymbol{\alpha}; P)|^{2s} \, \mathrm{d}\boldsymbol{\alpha}, \tag{1.7.1}$$

which by orthogonality counts the number of solutions of the system

$$\sum_{i=1}^{s} \left(x_i^j - x_{s+i}^j\right) = 0 \qquad (1 \leq j \leq k), \tag{1.7.2}$$

with $1 \leq \mathbf{x} \leq P$.

The problem of bounding the mean value $J_{s,k}(P)$ is known as Vinogradov's mean value theorem. One may consider the family of trivial solutions of system (1.7.2) where we suppose that $\{x_1, \ldots, x_s\} = \{x_{s+1}, \ldots, x_{2s}\}$. Hence, we certainly have that $J_{s,k}(P) \gg P^s$. Furthermore, by considering the contribution in the integral (1.7.1) from $\boldsymbol{\alpha}$ with $|\alpha_j| \leq \frac{1}{8k} P^{-j}$ $(1 \leq j \leq k)$ we deduce that $J_{s,k}(P) \gg P^{2s-\frac{1}{2}k(k+1)}$. Therefore one has that

$$J_{s,k}(P) \gg P^s + P^{2s-\frac{1}{2}k(k+1)}.$$

The Main Conjecture (now a theorem) in Vinogradov's mean value theorem, states that the above lower bound reflects the true order of magnitude of $J_{s,k}(P)$.

Theorem 1.7.1 (Main Conjecture in Vinogradov's mean value theorem – [92], [11]). *For any natural numbers s, k and any fixed $\epsilon > 0$ one has*

$$J_{s,k}(P) \ll P^{s+\epsilon} + P^{2s-\frac{1}{2}k(k+1)+\epsilon}. \tag{1.7.3}$$

Going a step further, the conjecture predicts that the estimate (1.7.3) holds with $\epsilon = 0$. However, this has not been proven yet. The case $k = 1$ is trivial. So is the case $k = 2$. The latter can be readily seen by using the identity

$$(a+b-c)^2 - (a^2+b^2-c^2) = 2(a-b)(a-c),$$

together with the estimate $d(n) \ll n^\epsilon \quad (n \to \infty)$, where $d(n) = \sum_{d|n} 1$ is the number of divisors of n. The first non-trivial case is $k = 3$. For $k = 3$ the inequality (1.7.3) was proven by Wooley in [92] using his efficient congruencing method, introduced in the breakthrough papers [88], [89]. This method was developed further by Wooley in a series of papers. For a summary of the main ideas and various applications of Vinogradov's mean value theorem, one may look at Wooley's ICM address [90] from 2014.

Parallel to Wooley's work, another breakthrough was taking place in the field of harmonic analysis with the proof of the ℓ^2-decoupling conjecture by Bourgain and Demeter [10]. Building on these ideas, Bourgain, Demeter and Guth [11] proved the estimate (1.7.3) for any $k \geq 4$, and hence established the Main Conjecture. Shortly after this paper, Wooley [95] established the same conclusion using his nested efficient congruencing method, which is a variant of the efficient congruencing method. In [95] it is also proven a number of results concerning "relatives" and applications of the Vinogradov's mean value theorem. A general discussion about the similarities between the decoupling theory and the efficient congruencing method can be found in the report of Pierce [60] for the Bourbaki seminar. For an exposition on decoupling theory and its applications one may look in the recent book of Demeter [35].

We now come to describe our result within the above context. Recall from (1.7.2) the system

$$\sum_{i=1}^{s} \left(x_i^j - x_{s+i}^j \right) = 0 \qquad (1 \leq j \leq k).$$

A key feature of this system is that it is a Translation-Dilation-Invariant (henceforth TDI) system. For $q \in \mathbb{N}$ and $\xi \in \mathbb{Z}$ and for each $1 \leq j \leq k$ one has that

$$\sum_{i=1}^{s} \left((qx_i - \xi + \xi)^j - (qx_{s+i} - \xi + \xi)^j \right) = \sum_{\ell=1}^{j} \binom{j}{\ell} \xi^{j-\ell} \sum_{i=1}^{s} \left((qx_i - \xi)^j - (qx_{s+i} - \xi)^j \right).$$

So for $\xi \neq 0$ and a fixed tuple $\mathbf{x}$ one can see that the left hand side of the above equality vanishes if and only if the right hand side vanishes. This property does not hold if one of the exponents

j is non-integral. Suppose that $\theta > 2$ is real and non-integral. For $\alpha \in \mathbb{R}$ we write

$$f_\theta(\alpha; P) = \sum_{1 \le x \le P} e(\alpha x^\theta).$$

In Chapter §2 we obtain an essentially optimal mean value estimate for the exponential sum $f_\theta(\alpha; P)$. Our result reads as follows.

Theorem 1.7.2. *Suppose that $\theta > 2$ is real and non-integral, and that $\kappa \ge 1$ is a real number. Suppose further that $s \ge \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ is a natural number. Then for any fixed $\epsilon > 0$, one has*

$$\int_{-\kappa}^{\kappa} |f_\theta(\alpha; P)|^{2s} \, \mathrm{d}\alpha \ll \kappa P^{2s-\theta+\epsilon}.$$

We emphasize here, that the implicit constant depends on ϵ, θ, and s, but not on κ and P. Furthermore, for $s > \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ one can take $\epsilon = 0$.

It will be apparent from the proof of the theorem that with slight modifications our argument works also if we consider the exponential sum

$$f_\theta(\boldsymbol{\alpha}; P) = \sum_{1 \le x \le P} e\left(\alpha_1 x + \cdots + \alpha_n x^n + \alpha_\theta x^\theta\right),$$

where $n = \lfloor \theta \rfloor$ and $\boldsymbol{\alpha} \in \mathbb{R}^{n+1}$. The details of this argument are presented in the Appendix B.

Theorem 1.7.3. *Suppose that $\theta > 2$ is real and non-integral, and write $n = \lfloor \theta \rfloor$. Let $\kappa \ge 1$ be a real number. Suppose further that $s \ge \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ is a natural number. Then for any fixed $\epsilon > 0$, one has*

$$\int_{-\kappa}^{\kappa} \int_{[0,1)^n} |f_\theta(\boldsymbol{\alpha}; P)|^{2s} \, \mathrm{d}\boldsymbol{\alpha} \ll \kappa P^{2s-\theta+\epsilon}.$$

We emphasize here, that the implicit constant depends on ϵ, θ, and s, but not on κ and P. Furthermore, for $s > \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ one can take $\epsilon = 0$.

We believe that one should be able to prove a discrete restriction estimate for the exponential sum $f_\theta(\boldsymbol{\alpha}; P)$ as in [94, Theorem 1.1]. This would require to modify some of the parts of the argument presented in the Appendix B. We leave this as a remark for a possible future work. Moreover, the argument given in the Appendix B seems to apply as well when the term x^θ in the exponential sum $f_\theta(\boldsymbol{\alpha}; P)$ is replaced by a generalised polynomial with leading term x^θ. Again we leave this as a remark for a possible future work.

1.8 The Davenport–Heilbronn–Freeman method

In this subsection we shall give a brief overview of the Davenport–Heilbronn method. We follow the exposition of [78, Chapter 11]. Subsequently we shall sketch the innovation introduced by Freeman in [36]. Our exposition about Freeman's variant follows Wooley's paper [86]. We shall focus in proving an asymptotic lower bound. For the asymptotic formula one may use the kernel functions of Freeman [39, Lemma 1] or apply a squeezing (sandwich) argument as in [20].

Fix integers $k \ge 2$ and $s \ge 2^k + 1$. Suppose that $\lambda_1, \ldots, \lambda_s$ are non-zero real numbers, not all of the same sign and not all in rational ratio. Let P be a large positive real number. We write

$N(P)$ to denote the number of integer solutions of the inequality

$$|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k| < 1$$

with $|\mathbf{x}| \leq P$, where $|\mathbf{x}| = \max_i |x_i|$. If necessary one may relabel the variables, so that from now we assume that λ_1/λ_2 is negative and irrational. Note that there is no loss of generality in assuming that the coefficients λ_i are not all of the same sign even if k is an odd positive integer. In such a case, one may replace if necessary, x_1^k by $(-x_1)^k$ and then relabel further.

The method of Davenport and Heilbronn uses a Fourier transform over the real line $\mathbb{R}$. The idea is to use an even kernel function $K : \mathbb{R} \to \mathbb{R}$ that allows us to make a weighted count for the number of solutions of the inequality in order to give a lower bound for the quantity $N(P)$. Define the function

$$K(\alpha) = \begin{cases} \left(\dfrac{\sin(\pi\alpha)}{\pi\alpha}\right)^2, & \text{when } \alpha \neq 0, \\ 1, & \text{when } \alpha = 0. \end{cases}$$

One can easily verify (see for example [30, Lemma 4]) that the Fourier transform of this function has the property that for any real ξ it satisfies

$$\int_{-\infty}^{\infty} K(\alpha) e(\alpha\xi) \mathrm{d}\alpha = \max\{0, 1 - |\xi|\}.$$

For $\alpha \in \mathbb{R}$ we define the generating function

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k),$$

and write $f_i(\alpha) = f(\lambda_i \alpha)\ (1 \leq i \leq s)$. One has

$$N(P) \gg \int_{-\infty}^{\infty} f_1(\alpha) \cdots f_s(\alpha) K(\alpha) \mathrm{d}\alpha. \tag{1.8.1}$$

For sufficiently small positive real numbers δ and ω we put

$$S_1(P) = P^{\delta} \qquad \text{and} \qquad T_1(P) = P^{\omega}.$$

One may now dissect the real line intro three disjoint subsets called the major, minor and trivial arcs, defined respectively by

$$\mathfrak{M} = \left\{\alpha \in \mathbb{R} : \ |\alpha| < S_1(P) P^{-k}\right\},$$

$$\mathfrak{m} = \left\{\alpha \in \mathbb{R} : \ S_1(P) P^{-k} \leq |\alpha| < T_1(P)\right\},$$

$$\mathfrak{t} = \left\{\alpha \in \mathbb{R} : \ |\alpha| \geq T_1(P)\right\}.$$

In contrast to the Hardy–Littlewood method here we only have one major arc around $0 = 0/1$. Moreover, since we are dealing with real forms no local solubility conditions for finite places is required. Hence, in this context there is no singular series.

The disposal of the trivial arcs is straightforward. Here we use Hua's inequality, appearing first in [45], which states that for any $1 \le j \le k$ and any fixed $\epsilon > 0$ one has

$$\int_0^1 |f(\alpha)|^{2^j} \, \mathrm{d}\alpha \ll P^{2^j - k + \epsilon}.$$

Moreover, we make use of the decay of the kernel function $K(\alpha)$, which for all α satisfies

$$|K(\alpha)| \ll \min\{1, |\alpha|^{-2}\}.$$

Splitting the set $\mathfrak{t}$ into unit intervals and using Hua's inequality we obtain

$$\int_{|\alpha| \ge T_1(P)} |f_i(\alpha)|^{2^k} \, K(\alpha) \mathrm{d}\alpha = o\left(P^{s-k}\right) \qquad (1 \le i \le s).$$

An application of Hölder's inequality then delivers

$$\int_{\mathfrak{t}} |f_1(\alpha) \cdots f_s(\alpha) K(\alpha)| \, \mathrm{d}\alpha = o\left(P^{s-k}\right). \tag{1.8.2}$$

Over the set of major arcs one wishes to compare the exponential sum $f_i(\alpha)$ with its continuous analogue $\upsilon_i(\alpha)$ defined by

$$\upsilon_i(\alpha) = \int_0^P e(\lambda_i \alpha x^k) \mathrm{d}\alpha \qquad (1 \le i \le s).$$

For $\alpha \in \mathfrak{M}$ as a consequence of Poisson's summation formula one has

$$f_i(\alpha) - \upsilon_i(\alpha) = O\left(P^{\delta}\right) \qquad (1 \le i \le s).$$

One may now use a standard telescoping summation argument to deduce that

$$\int_{\mathfrak{M}} f_1(\alpha) \cdots f_s(\alpha) K(\alpha) \mathrm{d}\alpha - \int_{-\infty}^{\infty} \upsilon_1(\alpha) \cdots \upsilon_s(\alpha) K(\alpha) \mathrm{d}\alpha = o\left(P^{s-k}\right).$$

Due to the decay of the kernel $K(\alpha)$ the integral over $(-\infty, \infty)$, which is the singular integral of our problem, converges absolutely. Making now a change of variables to linearise the expression and using the fact that the λ_i are not all of the same sign, one may deduce by applying Fubini's theorem that

$$\int_{-\infty}^{\infty} \upsilon_1(\alpha) \cdots \upsilon_s(\alpha) K(\alpha) \mathrm{d}\alpha = \int_{[0,P]^s} \max\left\{0,\ 1 - \left|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k\right|\right\} \mathrm{d}\mathbf{x} \gg P^{s-k},$$

where d**x** stands for the s-dimensional Lebesgue measure $\mathrm{d}x_1 \cdots \mathrm{d}x_s$. A comparison of the last two asymptotic estimates yields

$$\int_{\mathfrak{M}} f_1(\alpha) \cdots f_s(\alpha) K(\alpha) \mathrm{d}\alpha \gg P^{s-k}. \tag{1.8.3}$$

We are now dealing with the set of minor arcs. Since we assume that the ratio λ_1/λ_2 is irrational, by Dirichlet's theorem on Diophantine approximation [2] there exist infinitely many coprime pairs $(a, q) \in \mathbb{Z} \times \mathbb{N}$ with

$$\left| \frac{\lambda_1}{\lambda_2} - \frac{a}{q} \right| < \frac{1}{q^2}. \tag{1.8.4}$$

Consider now a sequence of such pairs $(a_n, q_n)_{n\in\mathbb{N}}$ and set $P_n = q_n^2$. It is here that we essentially restrict P to take values from a specific sequence. Note that the terms of the sequence $(P_n)_{n\in\mathbb{N}}$ become arbitrarily large.

Next, we recall Weyl's inequality which appeared in 1916 in the influential paper [82]. Suppose that for a real number $\alpha \in \mathbb{R}$ one has $|\alpha - a/q| \leq 1/q^2$ for some coprime $a \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then for any fixed $\epsilon > 0$ one has that

$$|f(\alpha)| \ll P^{1+\epsilon} \left(\frac{1}{q} + \frac{1}{P} + \frac{q}{P^k} \right)^{1/2^{k-1}}.$$

With the above choice $P_n = q_n^2$ we may infer by using Weyl's inequality that there exists some $\delta > 0$ so that for some $j \in \{1, 2\}$ and any $\alpha \in \mathfrak{m}$ one has for any fixed $\epsilon > 0$ that

$$f_j(\alpha) \ll P_n^{1+\epsilon} q_j^{-1/2^{k-1}} + P_n^{1-\delta},$$

where the q_1 and q_2 satisfy (1.8.4). Hence there exists some $\delta' > 0$ such that for all $\alpha \in \mathfrak{m}$ one has

$$\min\{f_1(\alpha), f_2(\alpha)\} \ll P_n^{1-\delta'} = o(P_n). \tag{1.8.5}$$

Using again Hua's inequality one may now deduce that

$$\int_{\mathfrak{m}} |f_1(\alpha) \cdots f_s(\alpha) K(\alpha)| \mathrm{d}\alpha = o\left(P_n^{s-k}\right). \tag{1.8.6}$$

Putting together (1.8.2), (1.8.3), (1.8.6) and invoking (1.8.1) we have shown that there exists a sequence of integers $(P_n)_{n\in\mathbb{N}}$ with $P_n \to \infty$ as $n \to \infty$ such that $N(P_n) \gg \gamma P_n^{s-k}$ for some positive real number γ depending at most on λ_i, k and s.

Without restricting the values of P the estimate (1.8.5) does not necessarily hold. As a matter of fact, it is pointed out in [78, p. 170] that by choosing λ_1/λ_2 appropriately one may show

$$\limsup_{P\to\infty} \left(\frac{1}{P} \sup_{\alpha\in\mathfrak{m}} \min\{|f_1(\alpha)|, |f_2(\alpha)|\} \right) > 0.$$

This can be seen by taking λ_1/λ_2 to be a Liouville number. A real number x is called a Liouville number, if for every $n \in \mathbb{N}$ there exist infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $q > 1$ such that $0 < |x - p/q| < 1/q^n$. It turns out that Liouville's numbers are transcendental numbers that can be approximated very well by rational numbers.

In Freeman's work [36], two are the main ingredients in the minor arc analysis. An ϵ-free

[2] Dirichlet's approximation theorem in its simplest form is stated as follows. Let $\alpha \in \mathbb{R}$ and suppose that $X \geq 1$ is a real number. Then there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ and $1 \leq q \leq X$ such that $|\alpha - a/q| \leq 1/(qX)$. For a proof see for example [44, Theorem 36]. It is a consequence of this, see [44, Theorem 185], that if α is irrational then there exist infinitely many such pairs (p, q) which satisfy the inequality $|\alpha - p/q| < 1/q^2$.

mean value estimate of the shape

$$\int_0^1 |f(\alpha)|^s \mathrm{d}\alpha \ll P^{s-k},$$

and a weak type analogue of Weyl's inequality. The innovation introduced in [36] is such a Weyl type inequality, which works as a replacement for (1.8.5) without restricting the values of P. This inequality which was inspired by the work of Bentkus and Götze [4], may be formulated as follows. Suppose that $2 \leq S(P) \leq P$ is a function that increases to infinity as $P \to \infty$. Then there exists a function $T(P) \leq S(P)$ depending only on λ_1, λ_2 and $S(P)$, which increases monotonically to infinity as $P \to \infty$ and such that

$$\sup_{S(P)P^{-k} \leq |\alpha| < T(P)} |f_1(\alpha) f_2(\alpha)| \leq P^2 T(P)^{-1/2^{k+1}}. \tag{1.8.7}$$

One may now dissect the real line in the same fashion as before, whereas now we take $S_1(P) = S(P)$ and $T_1(P) = T(P)$. The analysis of the major and trivial arcs is almost identical to the previous treatment.

In the set of minor arcs $\mathfrak{m}$ we apply the classical Hardy–Littlewood method by dissecting into major and minor arcs. We put

$$\mathfrak{N} = \bigcup_{\substack{0 \leq a \leq q \leq S(P) \\ (a,q)=1}} \left\{ \alpha \in [0,1) : \ |q\alpha - a| \leq S(P) P^{-k} \right\},$$

and write $\mathfrak{n} = [0,1) \setminus \mathfrak{N}$. Using standard techniques due to Vaughan [78, Lemma 4.9 & Theorem 4.4] and [77, Theorem A] one may obtain (respectively) that

$$\int_{\mathfrak{N}} |f(\alpha)|^s \mathrm{d}\alpha \ll P^{s-k} \qquad \text{and} \qquad \int_{\mathfrak{n}} |f(\alpha)|^s \mathrm{d}\alpha \ll P^{s-k}. \tag{1.8.8}$$

Let us remark here that the first estimate is valid provided only $s > \max\{4, k+1\}$. For any $n \in \mathbb{R}$ we may then deduce that

$$\int_n^{n+1} |f_i(\alpha)|^s \mathrm{d}\alpha \ll P^{s-k}.$$

At this step we partition the real line into two disjoint sets defined by

$$\mathfrak{P} = \{\alpha \in \mathbb{R} : \ \lambda_1 \alpha \pmod 1 \in \mathfrak{N}\} \qquad \text{and} \qquad \mathfrak{p} = \mathbb{R} \setminus \mathfrak{P} = \{\alpha \in \mathbb{R} : \ \lambda_1 \alpha \pmod 1 \in \mathfrak{n}\}.$$

One may split the interval $\mathfrak{m}$ into unit intervals of the shape $[n, n+1]$ for $n \in \mathbb{R}$. Then, by taking into account (1.8.8) an application of Hölder's inequality yields

$$\int_{[n+1] \cap \mathfrak{p}} |f_1(\alpha) \cdots f_s(\alpha) K(\alpha)| \mathrm{d}\alpha \ll P^{s-k} T(P)^{-1/s} = o\left(P^{s-k}\right).$$

We are left to deal with the set $\mathfrak{m} \cap \mathfrak{P}$. Here we make use of the estimate (1.8.7). For any unit interval $[n, n+1] \subset \mathfrak{m}$ one has

$$\sup_{\alpha \in [n,n+1]} |f_1(\alpha) f_2(\alpha)| \le P^2 T(P)^{-1/2^{k+1}}.$$

Combining this inequality together with the estimates in (1.8.8), we obtain by an application of Hölder's inequality that

$$\int_{[n,n+1]\cap\mathfrak{P}} |f_1(\alpha) \cdots f_s(\alpha) K(\alpha)| \mathrm{d}\alpha \ll P^{s-k} T(P)^{-\delta} = o\left(P^{s-k}\right),$$

for some $\delta = \delta(s,k) > 0$. Here we make use of the assumption that $s > 2k$. One may now sum over all unit intervals $[n, n+1] \subset \mathfrak{m}$. Taking into account the decay of the kernel function we deduce that

$$\int_{\mathfrak{m}} |f_1(\alpha) \cdots f_s(\alpha) K(\alpha)| \mathrm{d}\alpha = o\left(P^{s-k}\right),$$

which is exactly what we were aiming for.

Chapter 2

Diophantine inequalities of fractional degree

The work in this chapter is based (with minor changes) on the author's paper [64].

2.1 Introduction

A central topic in analytic number theory with a long history and various applications is the study of solubility of Diophantine inequalities. In the present chapter we are concerned with diagonal Diophantine inequalities whose degree is a fractional power. Let us make this more precise. Suppose that $\theta > 2$ is real and non-integral, and suppose that s is a positive integer. Let $\lambda_1, \dots, \lambda_s$ be fixed non-zero real numbers, not all of the same sign. Consider the generalised polynomial

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}(x_1, \dots, x_s) = \lambda_1 x_1^\theta + \cdots + \lambda_s x_s^\theta. \tag{2.1.1}$$

Suppose that τ is a fixed positive real number. A first natural question one can pose is the following. Does the inequality

$$|\mathcal{F}(\mathbf{x})| < \tau \tag{2.1.2}$$

admit a solution $\mathbf{x} = (x_1, \dots, x_s)$ in positive integers? Note that the assumption that not all of the coefficients λ_i are of the same sign, is natural in order to study the solubility of inequality (2.1.2), for otherwise one always has $|\mathcal{F}(\mathbf{x})| \geq |\lambda_1| + \cdots + |\lambda_s|$, and thus it is clear that $|\mathcal{F}(\mathbf{x})|$ fails to take arbitrarily small values. In the case where (2.1.2) admits infinitely many solutions one could additionally ask for the distribution of them. To formulate this, take P to be an arbitrary large positive real number that eventually we let tend to infinity. With this parameter serving as a quantification measure for the size of solutions of (2.1.2) we write $\mathcal{N}_{s,\theta}^\tau(P) = \mathcal{N}_{s,\theta}^\tau(P; \boldsymbol{\lambda})$ for the number of positive integer solutions $\mathbf{x} = (x_1, \dots, x_s)$ of (2.1.2) with $x_i \in [1, P]$ $(1 \leq i \leq s)$. A standard heuristic argument suggests that one typically expects $\gg P^{s-\theta}$ such solutions to (2.1.2).

The main result of this chapter reads as follows.

Theorem 2.1.1. *Suppose that $\theta > 2$ is real and non-integral, and suppose further that $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) + 1$ is a natural number. Then as $P \to \infty$ one has*

$$\mathcal{N}^{\tau}_{s,\theta}(P) = 2\tau\Omega(s,\theta;\boldsymbol{\lambda})P^{s-\theta} + o\left(P^{s-\theta}\right), \tag{2.1.3}$$

where

$$\Omega(s,\theta;\boldsymbol{\lambda}) = \left(\frac{1}{\theta}\right)^s |\lambda_1 \cdots \lambda_s|^{-1/\theta} C(s,\theta;\boldsymbol{\lambda}) > 0$$

with

$$C(s,\theta;\boldsymbol{\lambda}) = \int_{\mathcal{U}} \left(-\sigma_s(\sigma_1\beta_1 + \cdots + \sigma_{s-1}\beta_{s-1})\right)^{1/\theta - 1} (\beta_1 \cdots \beta_{s-1})^{1/\theta-1} \mathrm{d}\boldsymbol{\beta},$$

where $\mathrm{d}\boldsymbol{\beta}$ here stands for $\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_{s-1}$, and $\sigma_i = \lambda_i/|\lambda_i|$, and $\mathcal{U}$ denotes the set of points of the box $[0,|\lambda_1|] \times \cdots \times [0,|\lambda_{s-1}|]$, satisfying the condition that

$$-\sigma_s(\sigma_1\beta_1 + \cdots + \sigma_{s-1}\beta_{s-1}) \in [0,|\lambda_s|].$$

In particular, the inequality (2.1.2) possesses a non-trivial positive integer solution.

As a first comment on the asymptotic formula (2.1.3), let us remark that, as will be apparent to experts, the positivity of the real number $C(s,\theta;\boldsymbol{\lambda})$ follows immediately from the fact that the σ_i are not all of the same sign. In the special case where $\theta \in \mathbb{Q}$ is a rational number greater than 2 one can obtain a "special" family of solutions as follows. Let us write $\theta = p/q \in \mathbb{Q}$ for some $p,q \in \mathbb{N}$ with $\gcd(p,q) = 1$. Take $x_i = y_i^q$ $(1 \leq i \leq s)$ where $y_i \in \mathbb{N}$ with $y_i \asymp Y \asymp P^{1/q}$. Then inequality (2.1.2) takes the shape $|\lambda_1 y_1^p + \cdots + \lambda_s y_s^p| < \tau$. When the number of variables s is large enough in terms of p, as for example in [36, Theorem 1], one has that the number of solutions of this last inequality is $\gg Y^{s-p}$. Thus, the number of solutions of the inequality (2.1.2) satisfies $\mathcal{N}^{\tau}_{s,\theta}(P) \gg \left(P^{1/q}\right)^{s-p}$. On the other hand, the number of solutions obtained in this way is $o\left(P^{s-p/q}\right)$, and so the number of solutions obtained is very small compared with what is expected.

The first to consider studying additive problems with non-integral exponents is Segal in the early 1930's. In the papers [69], [70] and [71], Segal studied Waring's problem with non-integral exponents, and additionally (phrased slightly different in his work) considered the problem of solubility of the inequality

$$|x_1^\theta + \cdots + x_s^\theta - \nu| < \epsilon,$$

with $\theta > 2$ real and non-integral and $0 < \epsilon < \nu^{-c(\theta)/\theta}$, where $0 < c(\theta) < 1$ is a fixed number, depending only on θ. For large values of ν Segal showed the existence of a solution $\mathbf{x} \in \mathbb{N}^s$, provided that we are given $s \geq s_0(\theta)$ variables, where $s_0(\theta) \approx \theta(\lfloor\theta\rfloor + 1)2^{\lfloor\theta\rfloor+1} + 1$. In Theorem 2.1.3 below we improve this.

For questions and results on the interface between the fields of Diophantine inequalities and Diophantine approximation the interested reader can refer to the monograph [2], which contains an exposition of some of the most pivotal results in that area, dating up to late 1980's.

A great body of work in the existing literature is concerned with counting solutions inside a bounded box of side length P to indefinite inequalities of the shape

$$|\lambda_1 x_1^d + \cdots + \lambda_s x_s^d| < \tau, \tag{2.1.4}$$

where $d \geq 2$ is a natural number, and at least one of the ratios λ_i/λ_j is irrational. This last irrationality assumption is necessary, for otherwise if all the coefficients are in rational ratio then one can clear out the denominators by multiplying with the least common multiple which would reduce the inequality to an equation over the integers. The latter has been a separate area of research since the birth of the Hardy - Littlewood circle method in the early 1920's. The problem of the solubility of inequalities of the shape (2.1.4) first appears in the literature with the seminal work of Davenport and Heilbronn [30] in 1946. In that paper the authors prove that every real indefinite diagonal quadratic form in $s = 5$ variables can take arbitrarily small values. Their method to prove that result, what now is called the Davenport–Heilbronn method, is a Fourier analytic one over the entire real line. It is important here to mention that the main theorem of [30] proves that there exist arbitrarily large values of the parameter P such that (2.1.4) with $d = 2$ and $s = 5$ is soluble with $x_i \in [1, P] \cap \mathbb{Z}$. More precisely, they prove their result for a sequence of arbitrarily large numbers P, that depends essentially on the continued fraction expansion of the irrational ratio λ_i/λ_j. Thus, their conclusion would apply to boxes of side length P whenever this parameter P is a term of that specific sequence of values. This dependence was removed only in the early 2000's by Freeman. Beginning with [36], Freeman introduced a variant of the Davenport-Heilbronn method motivated by methods developed in [4]. This allowed Freeman to show the existence of infinitely many non-trivial integer solutions in boxes of any sufficiently large side length P, given roughly $d \log d$ variables (when d is large). Later in [39], Freeman established for the first time an asymptotic formula for the number of integer solutions of (2.1.4) inside the box $[-P, P]^s$, provided we have $s \geq 2^d + 1$ variables. The results of [36] and [39] were refined by Wooley in [86]. Since we are not dealing with an inequality of positive integral degree d as in (2.1.4), we finish here our rather short tour amongst results concerning that problem. The interested reader is directed to the papers of Freeman and Wooley for a general discussion.

It is reasonable to expect that a conclusion as in Theorem 2.1.1 would remain valid if instead of a homogenous inequality as of the type (2.1.2) we count solutions to an inhomogenous inequality of the shape

$$|\mathcal{F}(\mathbf{x}) - L| < \tau, \tag{2.1.5}$$

with $\mathcal{F}$ as in (2.1.1) and L being a given real number. We write $\mathcal{N}_{s,\theta}^{\tau}(P; \boldsymbol{\lambda}, L)$ to denote the number of positive integer solutions $\mathbf{x}$ of the inequality (2.1.5) with $x_i \in [1, P]$ $(1 \leq i \leq s)$. Here, the generalised polynomial $\mathcal{F}$ could be either indefinite or definite. In the case where $\mathcal{F}$ is indefinite there is no restriction on the size of L. However, one has to take boxes with side length P being sufficiently large in terms of s, θ and the coefficients λ_i of $\mathcal{F}$. On the other hand, if $\mathcal{F}$ is positive definite then one has to assume that $L \asymp_{\boldsymbol{\lambda}} P^\theta$. Namely, there exist suitable positive constants $c(\boldsymbol{\lambda}), C(\boldsymbol{\lambda})$ such that L belongs to an interval of the shape $c(\boldsymbol{\lambda})P^\theta \leq L \leq C(\boldsymbol{\lambda})P^\theta$. As is to be expected, the counting function of such solutions satisfies the same kind of asymptotic formula as in Theorem 2.1.1. The minor adjustments of the proof are postponed until §2.7.

Theorem 2.1.2. *Suppose that $\mathcal{F}$ is indefinite and let L be a fixed real number. Suppose further that $\theta > 2$ is real and non-integral, and that $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) + 1$ is a natural number.*

Then as $P \to \infty$ one has

$$\mathcal{N}^{\tau}_{s,\theta}(P;\boldsymbol{\lambda},L) = 2\tau\Omega(s,\theta;\boldsymbol{\lambda})P^{s-\theta} + o\left(P^{s-\theta}\right),$$

where $\Omega(s,\theta;\boldsymbol{\lambda})$ is a positive real number depending only on s,θ and the coefficients λ_i.

One can refer to [58] for such a conclusion for linear forms over primes, and to [40] for a general result concerning additive inhomogenous inequalities of integral degree $d \geq 2$.

More interesting is the case where $\mathcal{F}$ is positive definite. In such a case the problem is reformulated as a problem of representing arbitrarily large numbers by the generalised polynomial $\mathcal{F}$. Instead of counting solutions inside a box, we can count solutions that represent an arbitrary large real number. That is to say, for a positive real number ν sufficiently large in terms of s,θ and the positive number τ, we ask how many positive integer solutions are possessed by the inequality

$$|\mathcal{F}(\mathbf{x}) - \nu| < \tau. \tag{2.1.6}$$

We write $\rho_s(\tau,\nu) = \rho_s(\tau,\nu;\boldsymbol{\lambda})$ to denote the number of positive integer solutions of (2.1.6). One anticipates $\rho_s(\tau,\nu)$ to be large when τ is fixed and ν is large. Our next result establishes an asymptotic formula for the counting function $\rho_s(\tau,\nu)$.

Theorem 2.1.3. *Suppose that $\theta > 2$ is real and non-integral, and that $\tau \in (0,1]$ is a fixed real number. Suppose further that $s \geq (\lfloor 2\theta\rfloor + 1)(\lfloor 2\theta\rfloor + 2) + 1$ is a natural number. Then as $\nu \to \infty$ one has*

$$\rho_s(\tau,\nu) = 2(\lambda_1\cdots\lambda_s)^{-1/\theta}\frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)}\tau\nu^{s/\theta-1} + o\left(\nu^{s/\theta-1}\right).$$

A word is in order regarding the conclusions of Theorems 2.1.2 and 2.1.3. Though they look similar there is an essential difference between these two conclusions. As we already mentioned, in the situation of Theorem 2.1.2 we count solutions of an inequality inside a box, while in the situation covered by Theorem 2.1.3 we aim to "represent" a large positive number by the generalised polynomial $\mathcal{F}$. This difference is reflected in the shape of the asymptotic formulae we establish. In the indefinite case we consider boxes of arbitrarily large side length P, while in the definite case covered by Theorem 2.1.3, the main term in the asymptotic formula is limited by the size of the real number ν we wish to represent, since there is a natural height restriction imposed on a solution $\mathbf{x}$. This last observation is straightforward. Suppose that λ_i are all positive and suppose that we aim to represent a real number $\nu \leq N$ where N is a positive large parameter. Choose now $P = 2(\lambda_1^{-1/\theta} + \cdots + \lambda_s^{-1/\theta} + 1)N^{1/\theta}$. Then for any solution $\mathbf{x}$ of (2.1.6) with $\nu \leq N$ one has $x_i \leq P \ (1 \leq i \leq s)$. As a remark, we draw the attention of the reader to the recent works of Chow [23], [24] and Biggs [5] for the problem of representing a number by shifts of dth-powers where $d \in \mathbb{N}$. That is to say, for τ a sufficiently large positive real number, they investigate the solubility of the inequality

$$|(x_1-\mu_1)^d + \cdots + (x_s-\mu_s)^d - \tau| < \eta,$$

in integers $x_i > \mu_i$, where μ_i are fixed real numbers with μ_1 being irrational and η being a positive real number.

From now on we focus on Theorem 2.1.1. It is possible even at this stage to illustrate the route we take to tackle the problem. Define the exponential sum $f(\alpha) = f(\alpha; P)$ via

$$f(\alpha; P) = \sum_{1\leq x\leq P} e(\alpha x^{\theta}).$$

As in Freeman's variant of the Davenport-Heilbronn method, we are seeking mean value estimates of the asymptotic shape $\int_0^1 |f(\alpha)|^s \mathrm{d}\alpha \ll P^{s-\theta}$. The key mean value estimate that does the heavy lifting in the proof of Theorem 2.1.1 is the following.

Theorem 2.1.4. *Suppose that $\theta > 2$ is real and non-integral and that $\kappa \geq 1$ is a real number. Suppose further that $t \geq \frac{1}{2}(\lfloor 2\theta\rfloor + 1)(\lfloor 2\theta\rfloor + 2)$ is a natural number. Then for any fixed $\epsilon > 0$ one has*

$$\int_{-\kappa}^{\kappa} |f(\alpha)|^{2t} \,\mathrm{d}\alpha \ll \kappa P^{2t-\theta+\epsilon}.$$

We emphasize here, that the implicit constant depends on ϵ, θ and t, but not on κ and P. Furthermore, for $t > \frac{1}{2}(\lfloor 2\theta\rfloor + 1)(\lfloor 2\theta\rfloor + 2)$ one can take $\epsilon = 0$.

Before we announce the final result of this chapter, we pause for a moment to comment on the number of variables needed to establish the mean value estimate in Theorem 2.1.4. As we explain at the end of §2.2, the number $\frac{1}{2}(\lfloor 2\theta\rfloor + 1)(\lfloor 2\theta\rfloor + 2)$ stems from an application of the Main Conjecture in Vinogradov's mean value theorem to a system of degree $k = \lfloor 2\theta\rfloor + 1$. As a measure of comparison, note that when $\theta = d$ is a natural number the latest developments in Vinogradov's mean value theorem by Wooley's Nested Efficient Congruencing method [95, Corollary 14.7] deliver the bound

$$\int_0^1 |f(\alpha)|^s \mathrm{d}\alpha \ll P^{s-d},$$

provided $s \geq s_0$ where

$$s_0 = d^2 - d + 2\lfloor\sqrt{2d+2}\rfloor - \theta(d), \tag{2.1.7}$$

with $\theta(d)$ defined via

$$\theta(d) = \begin{cases} 1, & \text{when} \quad 2d+2 \geq \lfloor\sqrt{2d+2}\rfloor^2 + \lfloor\sqrt{2d+2}\rfloor, \\ 2, & \text{when} \quad 2d+2 < \lfloor\sqrt{2d+2}\rfloor^2 + \lfloor\sqrt{2d+2}\rfloor. \end{cases}$$

Making use of the above mean value estimate combined with a Weyl type inequality as in [86, Lemma 2.3] one can show that $s_0 + 1$ variables suffice to establish the anticipated asymptotic formula for the counting function $\mathcal{N}^{\tau}_{s,d}(P; \boldsymbol{\lambda})$. Hitherto, in view of [21, Theorem 11.3] one had to take $s \geq 2d^2$ when d is large. Incorporating (2.1.7) into [21] reduces the number of variables needed to establish the asymptotic formula for $\mathcal{N}^{\tau}_{s,d}(P; \boldsymbol{\lambda})$ by a factor of 2.

We briefly mention here the following very interesting statistical result due to Brüdern and Dietmann. From a measure theoretic point of view, the anticipated asymptotic formula holds for almost all (admissible) real forms $\lambda_1 x_1^d + \cdots + \lambda_s x_s^d$, provided we have more than $2d$ variables. More precisely, in [20] it is proven that given $s > 2d$ variables then for almost all (in the sense of Lebesgue measure) admissible values of the coefficients, there exists a positive real number

$C(s, d; \boldsymbol{\lambda})$ such that for all sufficiently large P one has

$$\left|\mathcal{N}_{s,d}^{\tau}(P) - 2\tau C(s, d; \boldsymbol{\lambda}) P^{s-d}\right| < P^{s-d-8^{-2d}},$$

uniformly in $0 < \tau \leq 1$. It would be interesting to derive an analogue with the exponent d replaced by an arbitrary positive fractional number θ.

Lastly, we encounter a weighted version of Theorem 2.1.4. For a sequence of complex numbers $(\mathfrak{a}_x)_{x \in \mathbb{N}}$ we write $f_{\mathfrak{a}}(\alpha) = f_{\mathfrak{a}}(\alpha; P)$ to denote the weighted exponential sum

$$f_{\mathfrak{a}}(\alpha; P) = \sum_{1 \leq x \leq P} \mathfrak{a}_x e(\alpha x^{\theta}).$$

Motivated by [94] we seek for an inequality

$$\|f_{\mathfrak{a}}(\alpha; P)\|_{L^{2s}} \leq C_P \|\mathfrak{a}_x\|_{\ell^2},$$

with the real number C_P depending at P and being uniform in $(\mathfrak{a}_x)_{x \in \mathbb{N}}$. Due to the fractional nature of θ, it is reasonable to expect a connection with Diophatine inequalities of the format (2.1.5). To do so, one has to detect solutions of inequalities by means of an appropriate kernel function. For $\alpha \in \mathbb{R}$ we define the function

$$\operatorname{sinc}(\alpha) = \begin{cases} \dfrac{\sin(\pi\alpha)}{\pi\alpha}, & \text{when } \alpha \neq 0, \\ 1, & \text{when } \alpha = 0, \end{cases} \tag{2.1.8}$$

and set $K(\alpha) = \operatorname{sinc}^2(\alpha)$ as in [30]. Our result reads as follows.

Theorem 2.1.5. *Suppose that $\theta > 2$ is real and non-integral, and suppose further that $s \geq 2(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) + 2$ is a natural number. Then one has*

$$\int_{-\infty}^{\infty} |f_{\mathfrak{a}}(\alpha)|^{2s} K(\alpha) \mathrm{d}\alpha \ll P^{s-\theta} \left(\sum_{1 \leq x \leq P} |\mathfrak{a}_x|^2 \right)^s.$$

In order to establish Theorem 2.1.5 we apply an elementary argument and "double" the number of variables, aiming eventually to reduce to a Diophantine problem of representing a large positive real number by a generalised polynomial $\mathcal{F}$ of the shape (2.1.1). Thus, one would be able to make use of Theorem 2.1.3. This explains the fact that for the inequality recorded in Theorem 2.1.5, we use twice as many number of variables needed in Theorem 2.1.3. This is a "cheap" argument. With harder work one could possibly eliminate the factor 2 and half the number of variables needed. This requires more effort and is not the focus of this work. The trick of "doubling" the number of variables is a classical argument in harmonic analysis and goes back to at least Zygmund in his paper [96]. More recently, it was used by Bourgain in the papers [8], [9], on discrete periodic Strichartz estimates.

2.2 Set up and overview of the method

We follow Freeman [39] in making use of appropriate kernel functions that allow one to bound the counting function $\mathcal{N}^{\tau}_{s,\theta}(P)$ from above and below. We make use of the following technical lemma.

Lemma 2.2.1. *Fix a positive integer h. Let a and b be real numbers with $0 < a < b$. Then there is an even real function $K(\alpha) = K(\alpha; a, b)$ such that the function ψ defined by*

$$\psi(\xi) = \int_{\mathbb{R}} e(\xi\alpha)K(\alpha)d\alpha$$

satisfies

$$\psi(\xi) \begin{cases} \in [0,1] & \textit{for} \quad \xi \in \mathbb{R} \\ = 0 & \textit{for} \quad |\xi| \geq b \\ = 1 & \textit{for} \quad |\xi| \leq a. \end{cases} \tag{2.2.1}$$

Moreover, K satisfies the bound

$$K(\alpha) \ll_h \min\left\{b, |\alpha|^{-1}, |\alpha|^{-h-1}(b-a)^{-h}\right\}. \tag{2.2.2}$$

Proof. This is [39, Lemma 1]. □

Set $\widetilde{\tau} = \tau(\log P)^{-1}$. We can now define the following two kernel functions

$$K_-(\alpha) = K(\alpha; \tau - \widetilde{\tau}, \tau) \quad \text{and} \quad K_+(\alpha) = K(\alpha; \tau, \tau + \widetilde{\tau}). \tag{2.2.3}$$

Note that by (2.2.2) we have

$$K_{\pm}(\alpha) \ll_{\tau,h} \min\{1, |\alpha|^{-1}, (\log P)^h \, |\alpha|^{-h-1}\}. \tag{2.2.4}$$

The estimate (2.2.4) is essential in the disposal of the set of trivial arcs. We make use of this for a particular choice of h to be chosen at a later stage. We refer to K_+, K_- as the upper and lower kernel respectively. The Fourier transform of K_+ provides us with an upper bound for $\mathcal{N}^{\tau}_{s,\theta}(P)$ while the Fourier transform of K_- provides a lower bound. To see this, let us write $\chi_\tau(\xi)$ for the indicator function of the interval $(-\tau, \tau)$, namely

$$\chi_\tau(\xi) = \begin{cases} 1, & \text{when} \quad |\xi| < \tau, \\ 0, & \text{when} \quad |\xi| \geq \tau. \end{cases} \tag{2.2.5}$$

By (2.2.1) one has that

$$\chi_{\tau-\widetilde{\tau}}(\xi) \leq \int_{-\infty}^{\infty} e(\alpha\xi)K_-(\alpha)\mathrm{d}\alpha \leq \chi_\tau(\xi)$$

and

$$\chi_\tau(\xi) \leq \int_{-\infty}^{\infty} e(\alpha\xi)K_+(\alpha)\mathrm{d}\alpha \leq \chi_{\tau+\widetilde{\tau}}(\xi).$$

Consequently, one has

$$\int_{-\infty}^{\infty} e(\xi\alpha)K_{-}(\alpha)\mathrm{d}\alpha \leq \chi_{\tau}(\xi) \leq \int_{-\infty}^{\infty} e(\xi\alpha)K_{+}(\alpha)\mathrm{d}\alpha. \tag{2.2.6}$$

We take a moment to point out that the expression

$$\left|\int_{-\infty}^{\infty} e(\xi\alpha)K_{\pm}(\alpha)\mathrm{d}\alpha - \chi_{\tau}(\xi)\right| \tag{2.2.7}$$

is zero when $||\xi| - \tau| > \widetilde{\tau}$ and at most 1 for values of ξ such that $||\xi| - \tau| \leq \widetilde{\tau}$.

We are now equipped to explain how we sandwich the counting function $\mathcal{N}_{s,\theta}^{\tau}(P)$. Recall that

$$f(\alpha) = \sum_{1\leq x\leq P} e(\alpha x^{\theta}).$$

We write $f_i(\alpha) = f(\lambda_i\alpha)$ $(1 \leq i \leq s)$ and put

$$R_{\pm}(P) = \int_{-\infty}^{\infty} f_1(\alpha)\cdots f_s(\alpha)K_{\pm}(\alpha)\mathrm{d}\alpha. \tag{2.2.8}$$

Take now $\xi = \lambda_1 x_1^{\theta} + \cdots + \lambda_s x_s^{\theta}$ in (2.2.1). If we sum over $1 \leq \mathbf{x} \leq P$ and take into account (2.2.5) and (2.2.6), we obtain

$$R_{+}(P) \geq \sum_{\substack{1\leq \mathbf{x}\leq P\\ |\mathcal{F}(\mathbf{x})|<\tau}} 1 = \mathcal{N}_{s,\theta}^{\tau}(P),$$

and

$$R_{-}(P) \leq \sum_{\substack{1\leq \mathbf{x}\leq P\\ |\mathcal{F}(\mathbf{x})|<\tau}} 1 = \mathcal{N}_{s,\theta}^{\tau}(P).$$

Thus, we conclude that

$$R_{-}(P) \leq \mathcal{N}_{s,\theta}^{\tau}(P) \leq R_{+}(P).$$

From the inequality above it is clear that in order to establish an asymptotic formula for $\mathcal{N}_{s,\theta}^{\tau}(P)$ it suffices to obtain asymptotic formulae for the integrals $R_{\pm}(P)$ that are asymptotically equal.

We now fix some notation. Put $\gamma = \theta - \lfloor\theta\rfloor \in (0,1)$. We set

$$\delta_0 = 4^{-\theta} \quad \text{and} \quad \omega = \min\left\{\frac{1-\gamma}{12}, 5^{-100\theta}\right\}. \tag{2.2.9}$$

We dissect the real line into three disjoint subsets as follows.

(i) The major arc $\mathfrak{M}$ around 0 given by

$$\mathfrak{M} = \left\{\alpha \in \mathbb{R} : |\alpha| < P^{-\theta+\delta_0}\right\}.$$

(ii) The minor arcs $\mathfrak{m}$ given by

$$\mathfrak{m} = \left\{\alpha \in \mathbb{R} : P^{-\theta+\delta_0} \leq |\alpha| < P^{\omega}\right\}.$$

(iii) The trivial arcs $\mathfrak{t}$ given by
$$\mathfrak{t} = \{\alpha \in \mathbb{R} : |\alpha| \geq P^{\omega}\}.$$

For a Lebesgue measurable set $\mathcal{B} \subset \mathbb{R}$ we define

$$R_{\pm}(P;\mathcal{B}) = \int_{\mathcal{B}} f_1(\alpha)\cdots f_s(\alpha)K_{\pm}(\alpha)\mathrm{d}\alpha. \tag{2.2.10}$$

So, by (2.2.8) one has

$$R_{\pm}(P) = R_{\pm}(P;\mathfrak{M}) + R_{\pm}(P;\mathfrak{m}) + R_{\pm}(P;\mathfrak{t}). \tag{2.2.11}$$

We describe now the general philosophy underlying the Davenport - Heilbronn method and give a brief overview of the strategy we follow in order to prove Theorem 2.1.1.

The starting point is an analytic representation for the counting function, as in (2.2.11). We begin with the major arc $\mathfrak{M}$. Typically when integrating over $\mathfrak{M}$ one expects to obtain a contribution of order asymptotically equal to $P^{s-\theta}$, namely

$$\int_{\mathfrak{M}} f_1(\alpha)\cdots f_s(\alpha)K_{\pm}(\alpha)\mathrm{d}\alpha \asymp P^{s-\theta},$$

where the implied constants are positive. This reflects the observation that for small values of $|\alpha|$ the exponential sum $f(\alpha) = \sum_{1\leq x\leq P} e(\alpha x^{\theta})$ displays less cancellation. That is to say, the contribution from a small neighbourhood around 0 dominates, giving rise to the main term of the asymptotic formula in Theorem 2.1.1. The major arc analysis is rather classical. We present a treatment based on Fourier's inversion theorem, similar to that in [20, §5], which in turn is a simplification of the treatment presented in [39].

We proceed now to describe the treatment of the sets of minor and trivial arcs. As is to be expected, this constitutes the most challenging part along the way to proving Theorem 2.1.1. Here one typically expects to demonstrate a smaller overall contribution of size $o\left(P^{s-\theta}\right)$. Namely, we aim to show
$$\int_{\mathfrak{m}\cup\mathfrak{t}} |f_1(\alpha)\cdots f_s(\alpha)K_{\pm}(\alpha)|\mathrm{d}\alpha = o\left(P^{s-\theta}\right).$$
The method we employ here is motivated by our aim to find an alternative route that is less sensitive to the fact that we are dealing with exponential sums having a smooth and not polynomial phase. With the classical version of Weyl's inequality out of the game, we put our efforts into gaining an almost full saving from a Hua type estimate for the exponential sum $f(\alpha)$. With this aim in mind we can now bound the mean value $\int_{\mathfrak{m}} |f(\alpha)|^s\mathrm{d}\alpha$ for $s \geq 2t$ by noting that

$$\int_{\mathfrak{m}} |f(\alpha)|^s\mathrm{d}\alpha \ll \left(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\right)^{s-2t} \int_{\mathfrak{m}} |f(\alpha)|^{2t}\mathrm{d}\alpha, \tag{2.2.12}$$

where t is an appropriate positive integer. It may be helpful to the reader if we mention here that we seek to choose t so that

$$\int_{\mathfrak{m}} |f(\alpha)|^{2t}\mathrm{d}\alpha \ll P^{2t-\theta+\epsilon},$$

for any fixed $\epsilon > 0$. Note that in order to establish the asymptotic formula (2.1.3) one has to add one extra variable and take $s \geq 2t + 1$. We bound each factor in (2.2.12) separately as follows.

For the first factor in (2.2.12) it suffices to show that for some $\eta > 0$ one has

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll P^{1-\eta}.$$

This is achieved by means of van der Corput's k-th derivative test. The key here is to investigate the magnitude of the derivatives of the phase function $x \mapsto \alpha x^\theta$.

For the second factor we aim to obtain an upper bound of the shape $\int_{\mathfrak{m}} |f(\alpha)|^{2t} \mathrm{d}\alpha \ll P^{2t-\theta+\epsilon}$. This is precisely the content of Theorem 2.1.4, which in a sense plays the role of Hua's inequality and does the heavy lifting in the proof of Theorem 2.1.1. To demonstrate this we begin by observing that a mean value of the shape

$$\int_{-1/(2\delta)}^{1/(2\delta)} |f(\alpha)|^{2t} \mathrm{d}\alpha$$

counts asymptotically the number of integer solutions $1 \leq \mathbf{x} \leq P$ of the corresponding underlying inequality

$$|x_1^\theta + \cdots + x_t^\theta - x_{t+1}^\theta - \cdots - x_{2t}^\theta| < \delta,$$

where $\delta > 0$. The idea is to exploit the Taylor expansion of the function $x \mapsto x^\theta$ to obtain an approximately TDI system. This idea seems to appear first in the work of Arkhipov and Zhitkov [1]. The new system is composed of an inhomogeneous Vinogradov system and a linear inequality. The degree of the Vinogradov system is dictated by the number of terms we consider in the asymptotic expansion, and is the same as the number of variables of the linear inequality, which arises naturally when we collect the smaller terms of the Taylor expansion. As one suspects, the height condition on the variables in the inequality is imposed by the vector that has as components the inhomogeneous side of the Vinogradov system. We follow the original approach of Arkhipov and Zhitkov and consider a degree k expansion where $k = \lfloor 2\theta \rfloor + 1$. However, our treatment differs from that in [1, Lemma 3] in two aspects. Firstly, we encounter from the very beginning an exponential sum with a smooth phase, while in [1, Lemma 3] the authors deal with an exponential sum whose phase is the integer $\lfloor x^\theta \rfloor \in \mathbb{N}$. Secondly, and most important, our treatment is a refinement of that presented in [1, Lemma 3]. In the latter, the authors obtain an estimate which is $P^{1/2}$ away from the near optimal one. By contrast we establish an essentially optimal estimate in Theorem 2.1.4.

Lastly, a word concerning the set of trivial arcs. To deal with the set of trivial arcs, we split the range $|\alpha| \geq P^\omega$ into dyadic intervals of the shape $(2^j, 2^{j+1}]$, and use Theorem 2.1.4 to obtain a mean value estimate over such interval. The decay of the kernel functions $K_\pm$ provides us with the necessary savings in the final summation.

2.3 An auxiliary mean value estimate

In this section we prove Theorem 2.1.4. To do so, we first collect some auxiliary results that we need in our proof.

For $t, k \in \mathbb{N}$ we define the mean value

$$J_{t,k}(P) = \int_{[0,1)^k} \left| \sum_{1 \le x \le P} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2t} \mathrm{d}\boldsymbol{\alpha}.$$

By orthogonality, one has that $J_{t,k}(P)$ counts the number of integer solutions of the system

$$\sum_{i=1}^{t} (x_i^j - x_{t+i}^j) = 0 \quad (1 \le j \le k),$$

with $\mathbf{x} \in [1, P]^{2t}$. The study of the mean value $J_{t,k}(P)$ goes back to the mid 1930's and Vinogradov [79]. The central problem here is to find upper bounds for $J_{t,k}(P)$. The Main Conjecture in Vinogradov's mean value theorem, now a theorem after the work of Wooley [92] for $k = 3$, and Bourgain, Demeter and Guth [11], for $k \ge 4$, reads as follows.

Theorem 2.3.1. *Suppose that $t \ge \frac{1}{2}k(k+1)$ is a natural number. Then for any fixed $\epsilon > 0$ one has*

$$J_{t,k}(P) \ll P^{t+\epsilon} + P^{2t - \frac{1}{2}k(k+1)}.$$

Proof. See [95, Corollary 1.3]. An estimate weaker by a factor P^ϵ can be found in [92, Theorem 1.1] for $k = 3$ and in [11, Theorem 1.1] for $k \ge 4$. □

In the proof of Theorem 2.1.4, we deal repeatedly with inequalities of the shape

$$|x_1^\theta + \cdots + x_t^\theta - x_{t+1}^\theta - \cdots - x_{2t}^\theta| < \delta, \tag{2.3.1}$$

where $\delta > 0$ is a fixed real number. Due to the fact that θ is not an integer, one cannot count directly the solutions via the usual orthogonality relation over the interval $[0, 1)$. As a surrogate, we make use of an auxiliary lemma which is a variant of [81, Lemma 2.1]. In order to state the lemma we first introduce some notation. Suppose that $I_1, I_2 \subset (0, \infty)$ are bounded intervals, and suppose further that $\mathcal{S} \subset (0, \infty)^2$ is a bounded set of lattice points. We write $V_t(I_1, I_2; \delta)$ to denote the number of integer solutions of inequality (2.3.1) with $x_1, x_{t+1} \in I_1$ and $x_i \in I_2$ for all $i \ne 1, t+1$. Similarly, we write $V_t(\mathcal{S}, I_2; \delta)$ to denote the number of integer solutions of inequality (2.3.1) with $(x_1, x_{t+1}) \in \mathcal{S}$ and $x_i \in I_2$ for all $i \ne 1, t+1$. For $\alpha \in \mathbb{R}$ and $i = 1, 2$ we put $H_i(\alpha) = H(\alpha; I_i)$, where

$$H(\alpha; I_i) = \sum_{x \in I_i} e(\alpha x^\theta).$$

Moreover, we write

$$H_{\mathcal{S}}(\alpha) = \sum_{(x_1, x_{t+1}) \in \mathcal{S}} e\left(\alpha(x_1^\theta - x_{t+1}^\theta)\right).$$

The lemma now reads as follows. We note here that if $I_1 = I_2$, then our result in (ii) is a special case of [81, Lemma 2.1] with $K = 1$ and $\omega = x^\theta$ in their notation.

Lemma 2.3.2. *Define the number Δ via the relation $2\Delta\delta = 1$.*

(i) One has

$$V_t(\mathcal{S}, I_2; \delta) \ll \delta \int_{-\Delta}^{\Delta} \left| H_{\mathcal{S}}(\alpha) H_2(\alpha)^{2t-2} \right| \mathrm{d}\alpha.$$

(ii) One has

$$\delta \int_{-\Delta}^{\Delta} \left| H_1(\alpha)^2 H_2(\alpha)^{2t-2} \right| \mathrm{d}\alpha \ll V_t(I_1, I_2; \delta) \ll \delta \int_{-\Delta}^{\Delta} \left| H_1(\alpha)^2 H_2(\alpha)^{2t-2} \right| \mathrm{d}\alpha.$$

The implicit constants in the above estimates are independent of $I_1, I_2, \mathcal{S}, \theta$ and δ.

Proof. The argument proceeds as in [81, Lemma 2.1]. For $x \in \mathbb{R}$ we define the functions

$$K(\alpha) = \mathrm{sinc}^2(\alpha) \qquad \text{and} \qquad \Lambda(x) = \max\{0, 1 - |x|\},$$

where recall from (2.1.8) the definition of the sinc function. It is well known, one may see for example in [29, Lemma 20.1], that for $x, \xi \in \mathbb{R}$ one has

$$K(\xi) = \int_{-\infty}^{\infty} e(-x\xi)\Lambda(x)\mathrm{d}x \quad \text{and} \quad \Lambda(x) = \int_{-\infty}^{\infty} e(x\xi)K(\xi)\mathrm{d}\xi. \tag{2.3.2}$$

We make use of Jordan's inequality, which states that for $0 < x \leq \frac{\pi}{2}$ one has

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1,$$

where the equality holds only if $x = \pi/2$. For a proof of this inequality see [48, p. 33]. Note here that for $|\alpha| < \frac{1}{2}$ one has $K(\alpha) > 4/\pi^2$.

For ease of notation we set

$$\sigma_{t,\theta}(\mathbf{x}) = x_1^\theta + \cdots - x_{2t}^\theta \qquad \text{and} \qquad \xi = \frac{1}{2\delta}\sigma_{t,\theta}(\mathbf{x}).$$

We first prove the upper bound in (i). Let $\mathbf{x}$ be a tuple counted by $V_t\left(\mathcal{S}, I_2; \delta\right)$. By Jordan's inequality one has

$$\frac{\pi^2}{4} K(\xi) > 1.$$

Hence

$$V_t\left(\mathcal{S}, I_2; \delta\right) \leq \frac{\pi^2}{4} \sum_{\mathbf{x}} K(\xi),$$

where the summation is over tuples $\mathbf{x}$ with $(x_1, x_{t+1}) \in \mathcal{S}$ and $x_i \in I_2 \ \ (i \neq 1, t+1)$. Using now (2.3.2) and making a change of variables by setting $u = 2\delta\alpha$ we successively obtain

$$V_t\left(\mathcal{S}, I_2; \delta\right) \leq \frac{\pi^2}{4} \sum_{\mathbf{x}} \int_{-\infty}^{\infty} e(u\xi)\Lambda(-u)\mathrm{d}u = \frac{\pi^2\delta}{2} \sum_{\mathbf{x}} \int_{-\infty}^{\infty} e(\alpha\sigma_{t,\theta}(\mathbf{x}))\Lambda(-2\delta\alpha)\mathrm{d}\alpha.$$

One can interchange the order of integration with that of summation. This is valid since the integral is absolutely convergent and we have a finite sum. Note here that

$$\sum_{\mathbf{x}} e(\alpha\sigma_{t,\theta}(\mathbf{x})) = H_{\mathcal{S}}(\alpha)H_2(\alpha)^{2t-2}.$$

Moreover, for $|\alpha| > \frac{1}{2\delta}$ one has $\Lambda(-2\delta\alpha) = 0$. Hence by the triangle inequality we conclude that

$$V_t(\mathcal{S}, I_2; \delta) \leq \frac{\pi^2\delta}{2} \int_{-\infty}^{\infty} \left| H_{\mathcal{S}}(\alpha) H_2(\alpha)^{2t-2} \right| \Lambda(-2\delta\alpha) \mathrm{d}\alpha$$

$$\ll \delta \int_{-\Delta}^{\Delta} \left| H_{\mathcal{S}}(\alpha) H_2(\alpha)^{2t-2} \right| \mathrm{d}\alpha.$$

Next we prove (ii). In order to establish the upper bound one may argue as in (i), whereas now we make use of the product $H_1(\alpha)^2 H_2(\alpha)^{2t-2}$. We give the proof of the lower bound. Let **x** be a tuple counted by $V_t(I_1, I_2; \delta)$. Then one has

$$0 < \Lambda(2\xi) < 1.$$

Thus, summing over **x** with $x_1, x_2 \in I_1$ and $x_i \in I_2$ $(i \neq 1, t+1)$, and using (2.3.2) we obtain

$$V_t(I_1, I_2; \delta) \geq \sum_{\mathbf{x}} \Lambda(2\xi).$$

Invoking again (2.3.2) and making a change of variables by setting $u = \delta\alpha$ we successively obtain

$$V_t(I_1, I_2; \delta) \geq \sum_{\mathbf{x}} \int_{-\infty}^{\infty} e(2u\xi) K(u) \mathrm{d}u = \delta \sum_{\mathbf{x}} \int_{-\infty}^{\infty} e(\alpha\sigma_{t,\theta}(\mathbf{x})) K(\delta\alpha) \mathrm{d}\alpha.$$

Since we assume that $x_1, x_{t+1} \in I_1$ one has

$$\sum_{\mathbf{x}} e(\alpha\sigma_{t,\theta}(\mathbf{x})) = \left| H_1(\alpha)^2 H_2(\alpha)^{2t-2} \right|.$$

Changing the order of summation and integration the preceding inequality now delivers

$$V_t(I_1, I_2; \delta) \geq \delta \int_{-\infty}^{\infty} \left| H_1(\alpha)^2 H_2(\alpha)^{2t-2} \right| K(\delta\alpha) \mathrm{d}\alpha. \tag{2.3.3}$$

Next, using again Jordan's inequality and the positivity of the integrand we obtain

$$\int_{-\infty}^{\infty} \left| H_1(\alpha)^2 H_2(\alpha)^{2t-2} \right| K(\delta\alpha) \mathrm{d}\alpha \geq \frac{4}{\pi^2} \int_{-\Delta}^{\Delta} \left| H_1(\alpha)^2 H_2(\alpha)^{2t-2} \right| \mathrm{d}\alpha.$$

Incorporating the above into (2.3.3) yields

$$V_t(I_1, I_2; \delta) \gg \delta \int_{-\Delta}^{\Delta} \left| H_1(\alpha)^2 H_2(\alpha)^{2t-2} \right| \mathrm{d}\alpha,$$

which completes the proof. □

From now on we set $k = \lfloor 2\theta \rfloor + 1$ and for $1 \leq j \leq k$ we define the binomial coefficients

$$b_j = \binom{\theta}{j} = \frac{\theta(\theta-1)\cdots(\theta-j+1)}{j\,!}.$$

For a tuple $\mathbf{h} = (h_1, \dots, h_k) \in \mathbb{Z}^k$ we write $\mathcal{H}(\mathbf{h}) = \mathcal{H}(h_1, \dots, h_k)$ to denote the expression

$$\mathcal{H}(h_1, \dots, h_k) = b_1 P^{\theta-1} h_1 + \cdots + b_k P^{\theta-k} h_k. \tag{2.3.4}$$

Lemma 2.3.3. *Suppose that $\theta > 2$ is real and non-integral. Let $k = \lfloor 2\theta \rfloor + 1$, and let t be a given natural number. Suppose that $P \geq k^{2k}$ is a real number. We write $T(P)$ to denote the number of integer solutions of the inequality*

$$|\mathcal{H}(\mathbf{h})| \leq 2t$$

in the variables h_j, satisfying $|h_j| \leq tP^{j/2} \quad (1 \leq j \leq k)$. Then one has

$$T(P) \leq 4(8t)^k P^{\frac{k(k+1)}{4} - \theta + \frac{1}{2}}.$$

Proof. This is [1, Lemma 1]. □

For technical reasons it is more convenient to work with exponential sums over dyadic intervals. For a positive real number X we write $g(\alpha) = g(\alpha; P)$ to denote the exponential sum

$$g(\alpha; P) = \sum_{P < x \leq 2P} e(\alpha x^\theta). \tag{2.3.5}$$

We are now equipped to prove the key estimate of this chapter. The following is a variant of Theorem 2.1.4, where now we are dealing with the exponential sum g. One may recover Theorem 2.1.4 by splitting the interval $[1, P]$ into $O(\log P)$ dyadic intervals and then apply the theorem below.

Theorem 2.3.4. *Let $\kappa \geq 1$ be a real number. Suppose that $t \geq \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ is a natural number. Then for any fixed $\epsilon > 0$ one has*

$$\int_{-\kappa}^{\kappa} |g(\alpha)|^{2t} \, \mathrm{d}\alpha \ll \kappa P^{2t-\theta+\epsilon}.$$

We emphasize here, that the implicit constant depends on ϵ, θ, and t, but not on κ and P. Furthermore, for $t > \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ one can take $\epsilon = 0$.

Proof. We set $I = (P, 2P]$. Apply Lemma 2.3.2 with $I_1 = I_2 = I$ and $\delta = \frac{1}{2\kappa}$. So, one has

$$\frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |g(\alpha)|^{2t} \, \mathrm{d}\alpha \ll V_t\left(I; \frac{1}{2\kappa}\right), \tag{2.3.6}$$

where $V_t\left(I; \frac{1}{2\kappa}\right)$ denotes the number of integer solutions of the inequality

$$|x_1^\theta + \cdots + x_t^\theta - x_{t+1}^\theta - \cdots - x_{2t}^\theta| < \frac{1}{2\kappa},$$

with $P < \mathbf{x} \leq 2P$. Since $\kappa \geq 1$ we plainly have that

$$V_t\left(I; \frac{1}{2\kappa}\right) \leq V_t\left(I; \frac{1}{2}\right),$$

where $V_t\left(I;\frac{1}{2}\right)$ denotes the number of integer solutions of the inequality

$$|x_1^\theta+\cdots+x_t^\theta-x_{t+1}^\theta-\cdots-x_{2t}^\theta|<\frac{1}{2},$$

with $P<\mathbf{x}\leq 2P$. Hence by (2.3.6) we obtain that

$$\int_{-\kappa}^{\kappa}|g(\alpha)|^{2t}\,\mathrm{d}\alpha\ll\kappa V_t\left(I;\frac{1}{2}\right). \tag{2.3.7}$$

We define the interval

$$\widetilde{I}=(P,P+(\lfloor\sqrt{P}\rfloor+1)\sqrt{P}].$$

Note that $I\subset\widetilde{I}$. Moreover, for $\alpha\in\mathbb{R}$ we write

$$\widetilde{g}(a)=\sum_{x\in\widetilde{I}}e(\alpha x^\theta).$$

It is apparent that $V_t\left(I;\frac{1}{2}\right)$ is bounded above by the number of integer solutions of the inequality

$$\left|\sum_{i=1}^{t}(x_i^\theta-x_{t+i}^\theta)\right|<\frac{1}{2},$$

with $x_1,x_{t+1}\in I$ and $x_i,x_{t+i}\in\widetilde{I}$ $(i\neq 1,t+1)$. Denote this number by $V_t(I,\widetilde{I};\frac{1}{2})$.

Apply now Lemma 2.3.2 with $I_1=I$ and $I_2=\widetilde{I}$. So, one has

$$V_t\left(I,\widetilde{I};\frac{1}{2}\right)\ll\int_{-1}^{1}|g(\alpha)|^2|\widetilde{g}(\alpha)|^{2t-2}\mathrm{d}\alpha. \tag{2.3.8}$$

Putting together (2.3.8) and the fact that $V_t\left(I;\frac{1}{2}\right)\leq V_t\left(I,\widetilde{I};\frac{1}{2}\right)$ reveals that

$$V_t\left(I;\frac{1}{2}\right)\ll\int_{-1}^{1}|g(\alpha)|^2|\widetilde{g}(\alpha)|^{2t-2}\mathrm{d}\alpha. \tag{2.3.9}$$

Our aim now is to bound the mean value on the right hand side of (2.3.9). For a natural number $\ell\geq 1$ we write

$$P_\ell=P+(\ell-1)\sqrt{P}, \tag{2.3.10}$$

and set $\widetilde{I}_\ell=(P_\ell,P_{\ell+1}]$. Note that $\widetilde{I}_\ell$ forms a cover of the interval $\widetilde{I}$ consisting of subintervals of length $\sqrt{P}$. We record this in the following inclusion

$$I\subset\widetilde{I}\subset\bigcup_{\ell=1}^{\lfloor\sqrt{P}\rfloor+1}\widetilde{I}_\ell. \tag{2.3.11}$$

For $\alpha\in\mathbb{R}$ we now set

$$\widetilde{g}_\ell(\alpha)=\sum_{x\in\widetilde{I}_\ell}e(\alpha x^\theta).$$

Incorporating the exponential sum $\widetilde{g}_\ell(\alpha)$, we deduce by the triangle inequality followed by an application of Hölder's inequality that

$$\int_{-1}^{1}|g(\alpha)|^2|\widetilde{g}(\alpha)|^{2t-2}\mathrm{d}\alpha \leq \int_{-1}^{1}|g(\alpha)|^2\left(\sum_{\ell=1}^{\lfloor\sqrt{P}\rfloor+1}|\widetilde{g}_\ell(\alpha)|\right)^{2t-2}\mathrm{d}\alpha$$

$$\leq \left(\lfloor\sqrt{P}\rfloor+1\right)^{2t-3}\sum_{\ell=1}^{\lfloor\sqrt{P}\rfloor+1}\int_{-1}^{1}|g(\alpha)|^2\,|\widetilde{g}_\ell(\alpha)|^{2t-2}\,\mathrm{d}\alpha.$$

Invoking (2.3.9), we infer that for some ℓ_0 with $1 \leq \ell_0 \leq \lfloor\sqrt{P}\rfloor + 1$ one has

$$\begin{aligned} V_t\left(I;\frac{1}{2}\right) &\ll \left(\lfloor\sqrt{P}\rfloor+1\right)^{2t-2}\int_{-1}^{1}|g(\alpha)|^2\,|\widetilde{g}_{\ell_0}(\alpha)|^{2t-2}\,\mathrm{d}\alpha \\ &\ll P^{t-1}\int_{-1}^{1}|g(\alpha)|^2\,|\widetilde{g}_{\ell_0}(\alpha)|^{2t-2}\,\mathrm{d}\alpha. \end{aligned} \tag{2.3.12}$$

We now turn our attention to the mean value on the right hand side of (2.3.12). One can apply Lemma 2.3.2 with $I_1 = I$ and $I_2 = \widetilde{I}_{\ell_0}$. Then one has that

$$\int_{-1}^{1}|g(\alpha)|^2\,|\widetilde{g}_{\ell_0}(\alpha)|^{2t-2}\,\mathrm{d}\alpha \ll V_t\left(I,\widetilde{I}_{\ell_0};\frac{1}{2}\right), \tag{2.3.13}$$

where $V_t\left(I,\widetilde{I}_{\ell_0};\frac{1}{2}\right)$ denotes the number of integer solutions of the inequality

$$\left|x_1^\theta - x_{t+1}^\theta + \sum_{i=2}^{t}(x_i^\theta - x_{t+i}^\theta)\right| < \frac{1}{2}, \tag{2.3.14}$$

with $x_1, x_{t+1} \in I$ and $x_i \in \widetilde{I}_{\ell_0}$ $(i \neq 1, t+1)$.

Recall that $\widetilde{I}_{\ell_0} = (P_{\ell_0}, P_{\ell_0+1}]$, where $P_{\ell_0} = P + (\ell_0 - 1)\sqrt{P}$. For each index $i \neq 1, t+1$ we set

$$y_i = x_i - P_{\ell_0}.$$

Clearly one has $0 < y_i \leq \sqrt{P}$. Upon noting that $P \gg P_{\ell_0} \gg \sqrt{P}$, an application of the mean value theorem of differential calculus yields for each index $i \neq 1, t+1$, that

$$|x_i^\theta - x_{t+i}^\theta| = \left|(y_i + P_{\ell_0})^\theta - (y_{t+i} + P_{\ell_0})^\theta\right| \asymp P_{\ell_0}^{\theta-1}|y_i - y_{t+i}| \ll P^{\theta-1/2}.$$

By the triangle inequality, the above estimate leads to

$$\left|\sum_{\substack{i=2\\ x_i\in\widetilde{I}_\ell}}^{t}(x_i^\theta - x_{t+i}^\theta)\right| \ll P^{\theta-1/2}.$$

Invoking (2.3.14) we now have that $|x_1^\theta - x_{t+1}^\theta| \ll P^{\theta-1/2}$. On the other hand, an application of the mean value theorem of differential calculus yields $|x_1^\theta - x_{t+1}^\theta| \asymp |x_1 - x_{t+1}|P^{\theta-1}$. Thus, we can conclude that $|x_1 - x_{t+1}| \ll \sqrt{P}$. One can rewrite this asymptotic estimate in the shape $|x_1 - x_{t+1}| \leq C_1\sqrt{P}$, where $C_1 > 0$ is a real number that depends at most on t and θ. In view of this new constraint one can return to inequality (2.3.14) and count solutions subject to the constraints

$$x_1, x_{t+1} \in I, \quad |x_1 - x_{t+1}| \leq C_1\sqrt{P}, \quad \text{and} \quad x_i \in \widetilde{I}_{\ell_0} \ (i \neq 1, t+1). \tag{2.3.15}$$

The points x_1, x_{t+1} belong to the interval I. Recalling the inclusion (2.3.11) we have that there are indices ℓ_1 and ℓ_2 for which

$$P_{\ell_1} < x_1 \leq P_{\ell_1+1} \quad \text{and} \quad P_{\ell_2} < x_{t+1} \leq P_{\ell_2+1}.$$

Then, combining (2.3.15) with the definition (2.3.10) of P_ℓ and using the fact that for each index ℓ we have $P_{\ell+1} - P_\ell = \sqrt{P}$, one can deduce that

$$C_1\sqrt{P} \geq |x_1 - x_{t+1}| \geq |P_{\ell_1} - P_{\ell_2}| - \sqrt{P} \geq (|\ell_1 - \ell_2| - 1)\sqrt{P}.$$

From the above computation we obtain that $|\ell_1 - \ell_2| \leq C_1 + 1$.

We now bound from above the number of integer solutions of the inequality (2.3.14) under the constraint (2.3.15). To do so, we make use of appropriate generating functions. We write $\mathcal{S} \subset I \times I$ for the set of lattice points $x_1, x_{t+1} \in I$ which satisfy $|x_1 - x_{t+1}| \leq C_1\sqrt{P}$. By Lemma 2.3.2 with $\mathcal{S}$ as above, $I_2 = \widetilde{I}_{\ell_0}$ and $\delta = \frac{1}{2}$ one has

$$V_t\left(I, \widetilde{I}_{\ell_0}; \frac{1}{2}\right) \ll \int_{-1}^{1} |H_{\mathcal{S}}(\alpha)\widetilde{g}_{\ell_0}(\alpha)^{2t-2}| \mathrm{d}\alpha, \tag{2.3.16}$$

where

$$H_{\mathcal{S}}(\alpha) = \sum_{(x_1, x_{t+1}) \in \mathcal{S}} e(\alpha(x_1^\theta - x_{t+1}^\theta)).$$

One can tile $\mathcal{S} \subset I \times I$ by invoking the cover $\left(\widetilde{I}_\ell\right)_\ell$. Taking into account our previous conclusion that $|\ell_1 - \ell_2| \leq C_1 + 1$ we infer that

$$|H_{\mathcal{S}}(\alpha)| \ll \sum_{\ell_1=1}^{\lfloor\sqrt{P}\rfloor+1} \sum_{\substack{\ell_2=1 \\ |\ell_1-\ell_2| \leq C_1+1}}^{\lfloor\sqrt{P}\rfloor+1} |\widetilde{g}_{\ell_1}(\alpha)||\widetilde{g}_{\ell_2}(\alpha)|.$$

Hence, for some $1 \leq \ell_1, \ell_2 \leq \lfloor\sqrt{P}\rfloor + 1$ one has

$$|H_{\mathcal{S}}(\alpha)| \ll P^{\frac{1}{2}}|\widetilde{g}_{\ell_1}(\alpha)||\widetilde{g}_{\ell_2}(\alpha)|.$$

One can now bound above the right hand side of (2.3.16). So we infer that

$$V_t\left(I, \widetilde{I}_{\ell_0}; \frac{1}{2}\right) \ll P^{\frac{1}{2}} \int_{-1}^{1} |\widetilde{g}_{\ell_1}(\alpha)\widetilde{g}_{\ell_2}(\alpha)\widetilde{g}_{\ell_0}(\alpha)^{2t-2}| \mathrm{d}\alpha. \tag{2.3.17}$$

Invoking the elementary inequality $|z_1 \cdots z_n| \ll |z_1|^n + \cdots + |z_n|^n$, which is valid for all complex numbers, one has that

$$|\widetilde{g}_{\ell_1}(\alpha)\widetilde{g}_{\ell_2}(\alpha)\widetilde{g}_{\ell_0}(\alpha)^{2t-2}| \ll |\widetilde{g}_{\ell_1}(\alpha)|^{2t} + |\widetilde{g}_{\ell_2}(\alpha)|^{2t} + |\widetilde{g}_{\ell_0}(\alpha)|^{2t}.$$

Hence, (2.3.17) delivers the estimate

$$V_t\left(I,\widetilde{I}_{\ell_0};\frac{1}{2}\right) \ll P^{\frac{1}{2}}\int_{-1}^{1}|\widetilde{g}_\ell(\alpha)|^{2t}\mathrm{d}\alpha,$$

where ℓ is one of the indices ℓ_1,ℓ_2,ℓ_0. Incorporating this estimate into (2.3.13) and recalling (2.3.12), we deduce that

$$V_t\left(I;\frac{1}{2}\right) \ll P^{t-\frac{1}{2}}\int_{-1}^{1}|\widetilde{g}_\ell(\alpha)|^{2t}\mathrm{d}\alpha. \tag{2.3.18}$$

We emphasize here that our choice of $1\le \ell \le \lfloor\sqrt{P}\rfloor+1$ is now fixed.

In view of (2.3.7) our aim in the rest of the proof is to bound the mean value appearing on the right hand side of (2.3.18). Appealing to Lemma 2.3.2 with $I_1=I_2=\widetilde{I}_\ell$ and $\delta=\frac{1}{2}$ one has

$$\int_{-1}^{1}|\widetilde{g}_\ell(\alpha)|^{2t}\mathrm{d}\alpha \ll V_t\left(\widetilde{I}_\ell;\frac{1}{2}\right), \tag{2.3.19}$$

where $V_t\left(\widetilde{I}_\ell;\frac{1}{2}\right)$ denotes the number of integer solutions of the inequality

$$\left|x_1^\theta+\cdots+x_t^\theta-(x_{t+1}^\theta+\cdots+x_{2t}^\theta)\right|<\frac{1}{2}, \tag{2.3.20}$$

with $x_i\in\widetilde{I}_\ell$. From now on we essentially follow [1, Lemma 3]. We set $Q_\ell=\lfloor P_\ell\rfloor$ and define $y_i=x_i-Q_\ell$ $(1\le i\le 2t)$. Note that

$$0<y_i<\lfloor\sqrt{P}\rfloor+1<Q_\ell.$$

This observation is immediate since by the definitions of P_ℓ and Q_ℓ one has

$$0\le P_\ell-\lfloor P_\ell\rfloor<y_i\le P_{\ell+1}-\lfloor P_\ell\rfloor=P_\ell-\lfloor P_\ell\rfloor+\sqrt{P}\le 1+\sqrt{P}<Q_\ell. \tag{2.3.21}$$

Then inequality (2.3.20) takes the shape

$$\left|(y_1+Q_\ell)^\theta+\cdots+(y_t+Q_\ell)^\theta-(y_{t+1}+Q_\ell)^\theta-\cdots(y_{2t}+Q_\ell)^\theta\right|<\frac{1}{2},$$

or equivalently,

$$Q_\ell^\theta\left|\left(1+\frac{y_1}{Q_\ell}\right)^\theta+\cdots+\left(1+\frac{y_t}{Q_\ell}\right)^\theta-\left(1+\frac{y_{t+1}}{Q_\ell}\right)^\theta-\cdots-\left(1+\frac{y_{2t}}{Q_\ell}\right)^\theta\right|<\frac{1}{2}. \tag{2.3.22}$$

We consider the function $h:(0,1)\to\mathbb{R}$ with $h(z)=(1+z)^\theta$. Then, a Taylor expansion up to the $k=\lfloor 2\theta\rfloor+1$ term around the point $z_0=0$ yields

$$h(z)=h(0)+\sum_{j=1}^{k}\frac{h^{(j)}(0)}{j\,!}z^j+r_k(z)=1+\sum_{j=1}^{k}b_jz^j+r_k(z),$$

where recall that

$$b_j=\binom{\theta}{j}=\frac{\theta(\theta-1)\cdots(\theta-j+1)}{j\,!},$$

is the j-th combinatorial coefficient of the expansion. Here $r_k(z)$ denotes the remainder term. In Lagrange's form, the remainder term takes the shape

$$r_k(z) = \frac{h^{(k+1)}(c)}{(k+1)!} z^{k+1} = b_{k+1}(1+c)^{\theta-k-1} z^{k+1},$$

for some $0 < c < z$. Upon nothing that $1 + c > 1$ and $\theta - k - 1 < 0$ we obtain

$$|r_k(z)| \leq |b_{k+1}||z|^{k+1}, \tag{2.3.23}$$

For each index $1 \leq i \leq 2t$ we write $z_i = y_i/Q_\ell$. In view of (2.3.21), one has $0 < z_i < 1$. By (2.3.21) it follows that for P sufficiently large one has $0 < z_i < \frac{2}{\sqrt{P}}$. Indeed, this follows immediately upon writing

$$z_i = \frac{y_i}{Q_\ell} \leq \frac{\sqrt{P}+1}{\lfloor P_\ell \rfloor} < \frac{\sqrt{P}+1}{P_\ell - 1} < \frac{2}{\sqrt{P}}.$$

Thus, (2.3.23) with $z = z_i$ delivers the following upper bound for the error term

$$\begin{aligned} |r_k(z_i)| &\leq |b_{k+1}| \left(\frac{2}{\sqrt{P}} \right)^{k+1} \\ &= |b_{k+1}| 2^{k+1} P^{-\frac{k+1}{2}}. \end{aligned} \tag{2.3.24}$$

Expanding each term occurring in (2.3.22), we obtain that

$$\begin{aligned} Q_\ell^\theta \left(1 + \frac{y_i}{Q_\ell}\right)^\theta &= Q_\ell^\theta \left(1 + b_1 \left(\frac{y_i}{Q_\ell}\right) + \cdots + b_k \left(\frac{y_i}{Q_\ell}\right)^k + r_k \left(\frac{y_i}{Q_\ell}\right)\right) \\ &= Q_\ell^\theta + b_1 Q_\ell^{\theta-1} y_1 + \cdots + b_k Q_\ell^{\theta-k} y_i^k + Q_\ell^\theta r_k \left(\frac{y_i}{Q_\ell}\right). \end{aligned} \tag{2.3.25}$$

For large P one has $Q_\ell \leq 2P$. So, by (2.3.24) and since $k + 1 = \lfloor 2\theta \rfloor + 2 > 2\theta$, we infer that as $P \to \infty$ one has

$$\begin{aligned} \left| Q_\ell^\theta r_k \left(\frac{y_i}{Q_\ell}\right) \right| &\leq |b_{k+1}| 2^{k+1} P^{-\frac{k+1}{2}} (2P)^\theta \\ &= |b_{k+1}| 2^{k+1+\theta} P^{\theta - \frac{k+1}{2}} \\ &= o(1). \end{aligned}$$

Consequently, when P is large enough in terms of k one has for each index $1 \leq i \leq 2t$ that

$$\left| Q_\ell^\theta r_k \left(\frac{y_i}{Q_\ell}\right) \right| \leq \frac{1}{4t}. \tag{2.3.26}$$

Substituting the asymptotic expansion (2.3.25) into (2.3.22) and taking into account (2.3.26) together with the symmetry of the inequality, we deduce that the number of integer solutions of the inequality (2.3.22) is bounded above by the number of integer solutions of the inequality

$$\left|\sum_{j=1}^{k} b_j Q_\ell^{\theta-j}\left(y_1^j+\cdots+y_t^j-y_{t+1}^j-\cdots-y_{2t}^j\right)\right| < \frac{2t}{4t}+\frac{1}{2}=1.$$

Rearranging the terms in the summation on the left hand side of the above expression, we can rewrite the last inequality in an equivalent form as

$$\left|b_1 Q_\ell^{\theta-1}\sum_{i=1}^{t}(y_i-y_{t+i})+\cdots+b_k Q_\ell^{\theta-k}\sum_{i=1}^{t}(y_i^k-y_{t+i}^k)\right|<1. \tag{2.3.27}$$

The number of integer solutions of the inequality (2.3.27) with $0<y_i<1+\lfloor\sqrt{P}\rfloor$ is bounded above by the number of integer solutions of the system

$$\begin{cases}\left|b_1 Q_\ell^{\theta-1}h_1+\cdots+b_k Q_\ell^{\theta-k}h_k\right|<1\\[2ex] \displaystyle\sum_{i=1}^{t}(y_i^j-y_{t+i}^j)=h_j \quad (1\le j\le k)\end{cases} \tag{2.3.28}$$

with $0<y_i\le Y$ where $Y=1+\lfloor\sqrt{P}\rfloor$. We denote this counting function by $Z_{t,k}\left(Y;\mathbf{h}\right)$. Note that the integers h_j satisfy the relation $|h_j|\le tY^j \quad (1\le j\le k)$.

We write $J_{t,k}\left(Y;\mathbf{h}\right)$ to denote the number of integer solutions of the inhomogeneous Vinogradov system

$$\sum_{i=1}^{t}(y_i^j-y_{t+i}^j)=h_j \qquad\qquad (1\le j\le k),$$

with $0<y_i\le Y$. By orthogonality one has

$$J_{t,k}\left(Y;\mathbf{h}\right)=\int_{[0,1)^k}\left|\sum_{0<y_i\le Y} e(\alpha_1 y+\cdots+\alpha_k y^k)\right|^{2t} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\mathrm{d}\boldsymbol{\alpha},$$

where as usual $\boldsymbol{\alpha}\cdot\mathbf{h}$ stands for $\alpha_1h_1+\cdots+\alpha_kh_k$. By the triangle inequality and in view of Theorem 2.3.1 one has for any fixed $\epsilon>0$ that

$$J_{t,k}\left(Y;\mathbf{h}\right)\le J_{t,k}\left(Y\right)\ll Y^{2t-\frac{1}{2}k(k+1)+\epsilon}. \tag{2.3.29}$$

Recall now the definition (2.3.4) of the expression $\mathcal{H}$. Returning our attention to the system (2.3.28) we see that

$$Z_{t,k}\left(Y;\mathbf{h}\right)\ll\sum_{\substack{|h_j|\le tY^j\\1\le j\le k\\|\mathcal{H}(\mathbf{h})|<1}} J_{t,k}\left(Y;\mathbf{h}\right),$$

which by the triangle inequality leads to

$$Z_{t,k}\left(Y;\mathbf{h}\right)\ll J_{t,k}\left(Y\right)\sum_{\substack{|h_j|\le tY^j\\1\le j\le k\\|\mathcal{H}(\mathbf{h})|<1}} 1. \tag{2.3.30}$$

Recall now that $Y = 1 + \lfloor \sqrt{P} \rfloor \ll \sqrt{P}$. One may estimate the sum on the right hand side of (2.3.30) by using Lemma 2.3.3. Hence, appealing to (2.3.29) and Lemma 2.3.3 the estimate (2.3.30) now delivers

$$\begin{aligned} Z_{t,k}\left(Y;\mathbf{h}\right) &\ll Y^{2t-\frac{1}{2}k(k+1)+\epsilon} \cdot P^{\frac{1}{4}k(k+1)-\theta+\frac{1}{2}} \\ &\ll P^{t-\theta+\frac{1}{2}+\epsilon}. \end{aligned} \tag{2.3.31}$$

Putting together (2.3.31), (2.3.19), and (2.3.18) we deduce that

$$V_t\left(I;\frac{1}{2}\right) \ll P^{2t-\theta+\epsilon},$$

which in view of (2.3.7) completes the proof of the theorem. □

It is convenient for the rest of the analysis to have in hand an estimate for the dilated exponential sum $f_i(\alpha) = f(\lambda_i\alpha)$.

Corollary 2.3.5. *Let λ be a fixed real number. Suppose that κ is a real number such that $\kappa|\lambda| \geq 1$. Suppose further that $t \geq \frac{1}{2}\left(\lfloor 2\theta \rfloor + 1\right)\left(\lfloor 2\theta \rfloor + 2\right)$ is a natural number. Then for any fixed $\epsilon > 0$ one has that*

$$\int_{-\kappa}^{\kappa} \left|f(\lambda\alpha)\right|^{2t} \mathrm{d}\alpha \ll \kappa P^{2t-\theta+\epsilon}.$$

The implicit constant depends on ϵ, λ, t,and θ, but not on κ and P.

Proof. Using the fact that $f(-\alpha) = \overline{f(\alpha)}$ for all $\alpha \in \mathbb{R}$ and changing variables, we see that

$$\begin{aligned} \int_{0}^{\kappa} \left|f(\lambda\alpha)\right|^{2t} \mathrm{d}\alpha = \int_{0}^{\kappa} \left|f(|\lambda|\alpha)\right|^{2t} \mathrm{d}\alpha &= \frac{1}{|\lambda|}\int_{0}^{\kappa|\lambda|} \left|f(u)\right|^{2t} \mathrm{d}u \\ &\ll \frac{1}{|\lambda|}\int_{-\kappa|\lambda|}^{\kappa|\lambda|} \left|f(u)\right|^{2t} \mathrm{d}u. \end{aligned}$$

Invoking Theorem 2.1.4 we are now done. □

2.4 Minor arcs analysis

We begin the analysis of the analytical representation (2.2.11) with the contribution coming from the minor arcs. Recall that this set is given by

$$\mathfrak{m} = \left\{\alpha \in \mathbb{R} : P^{-\theta+\delta_0} \leq |\alpha| < P^{\omega}\right\}.$$

Define the intervals $\mathfrak{m}^+ = \left[P^{-\theta+\delta_0}, P^{\omega}\right)$ and $\mathfrak{m}^- = \left(-P^{\omega}, -P^{-\theta+\delta_0}\right]$ and note that $\mathfrak{m} = \mathfrak{m}^+ \cup \mathfrak{m}^-$. One has $f_i(-\alpha) = \overline{f_i(\alpha)}$ for all $\alpha \in \mathbb{R}$. Moreover, the kernel functions $K_\pm(\alpha)$ are real valued and even. Recall (2.2.10). By a change of variables one has

$$R_\pm\left(P;\mathfrak{m}^-\right) = \overline{R_\pm\left(P;\mathfrak{m}^+\right)}, \tag{2.4.1}$$

where $\overline{R_\pm(P;\mathfrak{m}^+)}$ stands for the complex conjugate. Therefore, it suffices to deal with the set $\mathfrak{m}^+$.

We make use of the following variant of van der Corput's k-th derivative test, for bounding exponential sums.

Lemma 2.4.1. *Let $q \geq 0$ be an integer. Suppose that $f : (X, 2X] \to \mathbb{R}$ is a function having continuous derivatives up to the $(q+2)$-th order in $(X, 2X]$. Suppose also there is some $F > 0$, such that for all $x \in (X, 2X]$ we have*

$$FX^{-r} \ll |f^{(r)}(x)| \ll FX^{-r}, \tag{2.4.2}$$

for $r = 1, 2, \dots, q+2$. Then we have

$$\sum_{X<x\leq 2X} e(f(x)) \ll F^{1/(2^{q+2}-2)} X^{1-(q+2)/(2^{q+2}-2)} + F^{-1}X,$$

with the implied constant depending only upon the implied constants in (2.4.2).

Proof. See [43, Theorem 2.9]. □

Recall that in (2.3.5) we defined the exponential sum $g(\alpha) = g(\alpha; P)$ by

$$g(\alpha; P) = \sum_{P<x\leq 2P} e(\alpha x^\theta).$$

We put $g_i(\alpha) = g(\lambda_i\alpha)$ $(1 \leq i \leq s)$. Below we give a crude non-trivial upper bound for the exponential sum $f_i(\alpha)$ when $\alpha \in \mathfrak{m}^+$. We emphasize here that one can certainly improve this estimate. However, for our purposes the saving we obtain is sufficient.

Lemma 2.4.2. *For each index $1 \leq i \leq s$ one has for any fixed $\epsilon > 0$ that*

$$\sup_{\alpha\in\mathfrak{m}^+} |f_i(\alpha)| \ll P^{1-4^{-\theta}+\epsilon}. \tag{2.4.3}$$

Proof. Fix an index i. It suffices to show that

$$\sup_{\alpha\in\mathfrak{m}^+} |g_i(\alpha)| \ll P^{1-4^{-\theta}}.$$

Then one may split the interval $[1, P]$ into $O(\log P)$ dyadic intervals and the desired conclusion follows.

We set $\phi(x) = \lambda_i\alpha x^\theta$. For each integer $r \geq 1$ one has $\phi^{(r)}(x) = C_r\alpha x^{\theta-r}$, where we put $C_r = \lambda_i\theta(\theta-1)\cdots(\theta-r+1)$. It is apparent that for $P < x \leq 2P$ one has

$$\left|\phi^{(r)}(x)\right| \asymp FP^{-r},$$

where $F = |C_r||\alpha|P^\theta$. For $\alpha \in \mathfrak{m}^+ = \left[P^{-\theta+\delta_0}, P^\omega\right)$ one has

$$|C_r|P^{\delta_0} \leq F < |C_r|P^{\theta+\omega}.$$

We apply Lemma 2.4.1 with $q = n$, where $n = \lfloor \theta \rfloor$ in the integer part of θ. This yields that for any $\alpha \in \mathfrak{m}^+$ one has

$$|g_i(\alpha; X)| \ll P^{1-\eta} + P^{1-\delta_0},$$

where

$$\eta = \frac{n+2-\theta-\omega}{2^{n+2}-2}.$$

Upon recalling (2.2.9) one may easily verify that $\eta > 4^{-\theta}$ which completes the proof. □

By (2.2.4) one has $|K_\pm(\alpha)| \ll 1$. An application of Hölder's inequality reveals that

$$\int_{\mathfrak{m}^+} |f_1(\alpha) \cdots f_s(\alpha) K_\pm(\alpha)| \mathrm{d}\alpha \ll \left(\int_{\mathfrak{m}^+} |f_1(\alpha)|^s \mathrm{d}\alpha \right)^{1/s} \cdots \left(\int_{\mathfrak{m}^+} |f_s(\alpha)|^s \mathrm{d}\alpha \right)^{1/s}.$$

We set $\kappa = P^\omega$. Note that for large enough P one has $P^\omega |\lambda_i| \geq 1$. Combining Corollary 2.3.5 and the upper bound recorded in (2.4.3) we deduce that for any fixed $\epsilon > 0$ one has

$$\begin{aligned} \int_{\mathfrak{m}^+} |f_i(\alpha)|^s \mathrm{d}\alpha &\ll \left(\sup_{\alpha \in \mathfrak{m}^+} |f_i(\alpha)| \right)^{s-2t} \int_{-P^\omega}^{P^\omega} |f_i(\alpha)|^{2t} \mathrm{d}\alpha \\ &\ll P^{s-\theta} \cdot P^{-4^{-\theta}(s-2t)+\omega+\epsilon(s-2t+1)}, \end{aligned}$$

provided that $s > 2t \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$. Choosing $\epsilon = 5^{-100\theta} > 0$ as we are at liberty to do, and recalling from (2.2.9) that $\omega \leq 5^{-100\theta}$, we infer that

$$\int_{\mathfrak{m}^+} |f_i(\alpha)|^s \mathrm{d}\alpha \ll P^{s-\theta} \cdot P^{-4^{-\theta}(s-2t)+4^{-2\theta}(s-2t+2)} = o\left(P^{s-\theta}\right).$$

In the light of (2.4.1) we have established the following.

Lemma 2.4.3. *One has*

$$\int_{\mathfrak{m}} |f_1(\alpha) \cdots f_s(\alpha) K_\pm(\alpha)| \mathrm{d}\alpha = o\left(P^{s-\theta}\right),$$

provided $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) + 1$.

2.5 Trivial arcs analysis

In this section we deal with the set of trivial arcs. Recall that this set is given by

$$\mathfrak{t} = \{\alpha \in \mathbb{R} : |\alpha| \geq P^\omega\}.$$

Define $\mathfrak{t}^+ = [P^\omega, \infty)$ and $\mathfrak{t}^- = (-\infty, -P^\omega]$ and note that $\mathfrak{t} = \mathfrak{t}^+ \cup \mathfrak{t}^-$. Recall (2.2.10). A change of variables as in §2.4 yields

$$R_\pm\left(P; \mathfrak{t}^-\right) = \overline{R_\pm\left(P; \mathfrak{t}^+\right)}. \tag{2.5.1}$$

Hence it suffices to deal with the set $\mathfrak{t}^+$. By (2.2.4) with $h = 1$ one has that

$$\int_{\mathfrak{t}^+} |f_1(\alpha)\cdots f_s(\alpha)K_\pm(\alpha)|\mathrm{d}\alpha \ll \sum_{j=\lfloor \omega \log_2 P\rfloor}^{\infty} \frac{(\log P)}{2^{2j}} \int_{2^j}^{2^{j+1}} |f_1(\alpha)\cdots f_s(\alpha)|\mathrm{d}\alpha. \qquad (2.5.2)$$

An application of Hölder's inequality yields

$$\int_{2^j}^{2^{j+1}} |f_1(\alpha)\cdots f_s(\alpha)|\mathrm{d}\alpha \ll \left(\prod_{i=1}^{s} \int_{2^j}^{2^{j+1}} |f_i(\alpha)|^s \mathrm{d}\alpha\right)^{1/s}. \qquad (2.5.3)$$

Define $s_0 = (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ and note that s_0 is even. In making the trivial estimate

$$|f_i(\alpha)| = O(P)$$

it follows that

$$\int_{2^j}^{2^{j+1}} |f_i(\alpha)|^s \mathrm{d}\alpha \ll P^{s-s_0} \int_{2^j}^{2^{j+1}} |f_i(\alpha)|^{s_0} \mathrm{d}\alpha,$$

provided $s \geq s_0$. For sufficiently large P and for $j \geq \lfloor \omega \log_2 P \rfloor + 1$ one has $2^{j+1}|\lambda_i| \geq 1$ for each index i. Invoking Corollary 2.3.5 the above estimate yields that for any fixed $\epsilon > 0$ one has

$$\int_{2^j}^{2^{j+1}} |f_i(\alpha)|^s \mathrm{d}\alpha \ll 2^{j+1} P^{s-\theta+\epsilon} \qquad (1 \leq i \leq s).$$

By (2.5.2) and (2.5.3) we infer that

$$\int_{\mathfrak{t}^+} |f_1(\alpha)\cdots f_s(\alpha)K_\pm(\alpha)|\mathrm{d}\alpha \ll P^{s-\theta+\epsilon} \sum_{j=\lfloor \omega \log_2 P\rfloor}^{\infty} \frac{1}{2^j}.$$

Clearly one has

$$\sum_{j=\lfloor \omega \log_2 P\rfloor}^{\infty} \frac{1}{2^j} \ll P^{-\omega}.$$

Hence by choosing $\epsilon = \frac{\omega}{2} > 0$ the previous estimate delivers

$$\int_{\mathfrak{t}^+} |f_1(\alpha)\cdots f_s(\alpha)K_\pm(\alpha)|\mathrm{d}\alpha \ll P^{s-\theta-\frac{\omega}{2}} = o\left(P^{s-\theta}\right).$$

In the light of (2.5.1) we have established the following.

Lemma 2.5.1. *One has*

$$\int_{\mathfrak{t}} |f_1(\alpha)\cdots f_s(\alpha)K_\pm(\alpha)|\mathrm{d}\alpha = o\left(P^{s-\theta}\right),$$

provided $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$.

2.6 Major arc analysis and the asymptotic formula

Now we deal with the contribution of the major arc

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| < P^{-\theta+\delta_0}\}$$

around zero. The corresponding analytical approximations for the generating functions $f_i(\alpha)$ are given by

$$v_i(\alpha) = v(\lambda_i\alpha) = \int_0^P e(\lambda_i\alpha\gamma^\theta)\mathrm{d}\gamma \qquad (1 \le i \le s). \tag{2.6.1}$$

An application of partial summation delivers

$$f_i(\alpha) - v_i(\alpha) = O\left(1 + P^\theta|\alpha|\right),$$

uniformly for $\alpha \in \mathbb{R}$. Thus for $\alpha \in \mathfrak{M}$ one has

$$f_i(\alpha) - v_i(\alpha) \ll P^{\delta_0}.$$

The above estimate in combination with the trivial bounds $|f_i(\alpha)|, |v_i(\alpha)| \le P$ and the telescoping sum

$$f_1 \cdots f_s - v_1 \cdots v_s = \sum_{i=1}^{s} f_1 \cdots f_{i-1}(f_i - v_i)v_{i+1} \cdots v_s,$$

reveals that for $\alpha \in \mathfrak{M}$ one has

$$f_1(\alpha) \cdots f_s(\alpha) - v_1(\alpha) \cdots v_s(\alpha) \ll O(P^{s-1+\delta_0}).$$

Integrating over $\mathfrak{M}$ yields

$$\begin{aligned}\int_{\mathfrak{M}} f_1(\alpha) \cdots f_s(\alpha)K_\pm(\alpha)\mathrm{d}\alpha - \int_{\mathfrak{M}} v_1(\alpha) \cdots v_s(\alpha)K_\pm(\alpha)\mathrm{d}\alpha &\ll \int_{\mathfrak{M}} P^{s-1+\delta_0}\mathrm{d}\alpha \\ &= P^{s-1+\delta_0}\text{meas}\,(\mathfrak{M}) \\ &\asymp P^{s-\theta-1+2\delta_0},\end{aligned} \tag{2.6.2}$$

where in the last step we used the fact meas $(\mathfrak{M}) \asymp P^{-\theta+\delta_0}$. By (2.2.4) one has $|K_\pm(\alpha)| \ll 1$. Using integration by parts one has

$$v_i(\alpha) \ll \min\{P, |\alpha|^{-1/\theta}\} \ll \frac{P}{\left(1 + P^\theta|\alpha|\right)^{1/\theta}} \qquad (1 \le i \le s). \tag{2.6.3}$$

So we deduce that

$$\int_{\mathbb{R}\setminus\mathfrak{M}} v_1(\alpha) \cdots v_s(\alpha)K_\pm(\alpha)\mathrm{d}\alpha \ll \int_{|\alpha|>P^{-\theta+\delta_0}} |\alpha|^{-s/\theta}\mathrm{d}\alpha \ll P^{s-\theta-\delta_0(s/\theta-1)}, \tag{2.6.4}$$

where in the last step we used the hypothesis $s > 2\theta$.

The singular integral of our problem is given by

$$\mathcal{I}_\pm = \int_{-\infty}^{\infty} \upsilon_1(\alpha)\cdots\upsilon_s(\alpha)K_\pm(\alpha)\mathrm{d}\alpha. \tag{2.6.5}$$

Note that by (2.2.4) and (2.6.3) the integral $\mathcal{I}_\pm$ is well defined and absolutely convergent. Combining (2.6.2) and (2.6.4) and since $2\delta_0 < 1$, we see that

$$\mathcal{I}_\pm = \int_{\mathfrak{M}} f_1(\alpha)\cdots f_s(\alpha)K_\pm(\alpha)\mathrm{d}\alpha + o\left(P^{s-\theta}\right). \tag{2.6.6}$$

For $\alpha \in \mathbb{R}$ we put

$$\Phi(\alpha) = \upsilon_1(\alpha)\cdots\upsilon_s(\alpha) = \int_{[0,P]^s} e\left(\alpha(\lambda_1\gamma_1^\theta + \cdots + \lambda_s\gamma_s^\theta)\right)\mathrm{d}\boldsymbol{\gamma}. \tag{2.6.7}$$

In view of (2.6.3) we see that Φ is an integrable function. Making a change of variables by putting $\gamma_i = P\left(\beta_i|\lambda_i|^{-1}\right)^{1/\theta}$ $(1 \le i \le s)$ yields

$$\Phi(\alpha) = \left(\frac{P}{\theta}\right)^s |\lambda_1\cdots\lambda_s|^{-1/\theta}\int_{\mathcal{B}}(\beta_1\cdots\beta_s)^{1/\theta-1}e\left(\alpha P^\theta(\sigma_1\beta_1+\cdots+\sigma_s\beta_s)\right)\mathrm{d}\boldsymbol{\beta}, \tag{2.6.8}$$

where $\sigma_i = \lambda_i/|\lambda_i| \in \{\pm 1\}$ are not all equal and $\mathcal{B} = [0,|\lambda_1|]\times\cdots\times[0,|\lambda_s|]$. Let $\widetilde{\beta} \in \mathbb{R}$ be a parameter. We now write $\mathcal{U}(\widetilde{\beta}) = \mathcal{U}(\widetilde{\beta};\boldsymbol{\lambda}) \subset \mathbb{R}^{s-1}$ for the domain defined through the linear inequalities

$$0 \le \beta_i \le |\lambda_i| \quad (1 \le i \le s-1), \qquad 0 \le \widetilde{\beta} - \sigma_s\sigma_1\beta_1 - \cdots - \sigma_s\sigma_{s-1}\beta_{s-1} \le |\lambda_s|.$$

We set

$$\Psi_0(\widetilde{\beta}) = \int_{\mathcal{U}(\widetilde{\beta})}\left(\widetilde{\beta} - \sigma_s\sigma_1\beta_1 - \cdots - \sigma_s\sigma_{s-1}\beta_{s-1}\right)^{1/\theta-1}(\beta_1\cdots\beta_{s-1})^{1/\theta-1}\mathrm{d}\beta_1\cdots\mathrm{d}\beta_{s-1}.$$

For $\widetilde{\beta} \in [0,|\lambda_s|]$ the map $\widetilde{\beta} \mapsto \Psi_0(\widetilde{\beta})$ defines a non-negative and continuous function. Put

$$\Psi(\widetilde{\beta}) = \begin{cases} \Psi_0(\widetilde{\beta}), & \text{if } \widetilde{\beta} \in [0,|\lambda_s|], \\ 0, & \text{otherwise.} \end{cases} \tag{2.6.9}$$

Note that $\Psi(\widetilde{\beta})$ is a non-negative and compactly supported function defined over $\mathbb{R}$, which has precisely two points of discontinuity, at $\widetilde{\beta} = 0, |\lambda_s|$. We set

$$\widetilde{\beta} = \beta_s + \sigma_1\sigma_s\beta_1 + \cdots + \sigma_{s-1}\sigma_s\beta_{s-1}.$$

Replace in (2.6.8) the variable β_s by $\widetilde{\beta}$. Letting now $\widetilde{\beta}$ vary through $\mathbb{R}$ and using the fact that $\Psi(\widetilde{\beta})$ is compactly supported we obtain

$$\Phi(\alpha) = \left(\frac{P}{\theta}\right)^s |\lambda_1\cdots\lambda_s|^{-1/\theta}\int_{-\infty}^{\infty}\Psi(\widetilde{\beta})e(\alpha P^\theta\sigma_s\widetilde{\beta})\mathrm{d}\widetilde{\beta}. \tag{2.6.10}$$

Since Φ and Ψ are integrable we may apply Fourier's inversion theorem. Together with a substitution that replaces α by $\alpha P^{-\theta}$ we obtain that

$$\Psi(\widetilde{\beta}) = \left(\frac{\theta}{P}\right)^s |\lambda_1 \cdots \lambda_s|^{1/\theta} \int_{-\infty}^{\infty} \Phi(\alpha P^{-\theta}) e(-\sigma_s \widetilde{\beta}\alpha) \mathrm{d}\alpha. \tag{2.6.11}$$

Putting together (2.6.5), (2.6.8) and (2.6.10), we infer that

$$\begin{aligned} \mathcal{I}_{\pm} &= \left(\frac{P}{\theta}\right)^s |\lambda_1 \cdots \lambda_s|^{-1/\theta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(\widetilde{\beta}) e(\alpha P^{\theta} \sigma_s \widetilde{\beta}) K_{\pm}(\alpha) \mathrm{d}\alpha \mathrm{d}\widetilde{\beta} \\ &= \left(\frac{P}{\theta}\right)^s |\lambda_1 \cdots \lambda_s|^{-1/\theta} \int_{-\infty}^{\infty} \Psi(\widetilde{\beta}) \left(\int_{-\infty}^{\infty} e(\alpha P^{\theta} \sigma_s \widetilde{\beta}) K_{\pm}(\alpha) \mathrm{d}\alpha \right) \mathrm{d}\widetilde{\beta}. \end{aligned} \tag{2.6.12}$$

By the comment following (2.2.7) one has

$$\int_{-\infty}^{\infty} e(\alpha P^{\theta} \sigma_s \widetilde{\beta}) K_{\pm}(\alpha) \mathrm{d}\alpha = \chi_{\tau}\left(P^{\theta} \sigma_s \widetilde{\beta}\right), \tag{2.6.13}$$

unless $\widetilde{\beta}$ satisfies the relation $\left| |P^{\theta} \sigma_s \widetilde{\beta}| - \tau \right| < \widetilde{\tau}$, where recall that we have set $\widetilde{\tau} = \tau (\log P)^{-1}$. The measure of the set of points $\widetilde{\beta}$ which satisfy the latter inequality is $O\left(\tau P^{-\theta} (\log P)^{-1}\right) = o\left(P^{s-\theta}\right)$. The contribution of such points to the integral is $o\left(P^{s-\theta}\right)$ and hence one may ignore this set. Therefore, we may assume from now on that (2.6.13) is valid. Recalling that $\chi_{\tau}(\cdot)$ denotes the characteristic function of the interval $(-\tau, \tau)$ we may rewrite (2.6.13) as

$$\int_{-\infty}^{\infty} e(\alpha P^{\theta} \sigma_s \widetilde{\beta}) K_{\pm}(\alpha) \mathrm{d}\alpha = \begin{cases} 1, & \text{if } |\widetilde{\beta}| < \tau P^{-\theta}, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{2.6.14}$$

Lemma 2.6.1. *For $|\widetilde{\beta}| < \tau P^{-\theta}$ one has*

$$\Psi(\widetilde{\beta}) - \Psi(0) \ll \tau P^{-\theta}.$$

Proof. By (2.6.7) and (2.6.3) one has

$$|\Phi(\alpha P^{-\theta})| = \prod_{i=1}^{s} \left| \upsilon_i(\alpha P^{-\theta}) \right| \ll \frac{P^s}{(1+|\alpha|)^{s/\theta}}.$$

Thus, by (2.6.11) one has

$$\begin{aligned} |\Psi(\widetilde{\beta}) - \Psi(0)| &\leq \left(\frac{\theta}{P}\right)^s |\lambda_1 \cdots \lambda_s|^{-1/\theta} \int_{-\infty}^{\infty} |\Phi(\alpha P^{-\theta})| |e(-\sigma_s \widetilde{\beta}\alpha) - 1| \mathrm{d}\alpha \\ &\ll \int_{-\infty}^{\infty} \frac{1}{(1+|\alpha|)^{s/\theta}} |e(-\sigma_s \widetilde{\beta}\alpha) - 1| \mathrm{d}\alpha. \end{aligned}$$

Note that for any $x \in \mathbb{R}$ one has

$$|e(x) - 1| \leq 2\pi |x|.$$

Using this inequality we deduce that

$$\Psi(\widetilde{\beta}) - \Psi(0) \ll |\widetilde{\beta}| \int_{-\infty}^{\infty} \frac{|\alpha|}{(1+|\alpha|)^{s/\theta}} \mathrm{d}\alpha \ll \tau P^{-\theta},$$

since for $s > 2\theta$ the integral with respect to α is absolutely convergent. □

We now return to (2.6.12) and substitute $\Psi(\widetilde{\beta}) = \Psi(0) + O(\tau P^{-\theta})$. In view of (2.6.14) this yields

$$\mathcal{I}_{\pm} = 2\tau\Omega(s,\theta;\boldsymbol{\lambda})P^{s-\theta} + O\left(P^{s-2\theta}\right), \tag{2.6.15}$$

where

$$\Omega(s,\theta;\boldsymbol{\lambda}) = \left(\frac{1}{\theta}\right)^s |\lambda_1 \cdots \lambda_s|^{-1/\theta} C(s,\theta;\boldsymbol{\lambda}) \ > 0,$$

and $C(s,\theta;\boldsymbol{\lambda}) = \Psi(0)$ with $\Psi(0)$ given by (2.6.9) so that

$$\Psi(0) = \int_{\mathcal{U}(0)} \left(-\sigma_s(\sigma_1\beta_1 + \cdots + \sigma_{s-1}\beta_{s-1})\right)^{1/\theta-1} (\beta_1 \cdots \beta_{s-1})^{1/\theta-1} \mathrm{d}\beta_1 \cdots \mathrm{d}\beta_{s-1}.$$

Note that $\Psi(0)$ is positive since not all of the σ_i are equal. This can be readily seen as follows.

Let $|\lambda_0| = \min_i |\lambda_i|$. Trivially one has

$$\Psi(0) \gg \int_0^{|\lambda_0|} \cdots \int_0^{|\lambda_0|} \left(-\sigma_s(\sigma_1\beta_1 + \cdots + \sigma_{s-1}\beta_{s-1})\right)^{1/\theta-1} (\beta_1 \cdots \beta_{s-1})^{1/\theta-1} \mathrm{d}\boldsymbol{\beta}. \tag{2.6.16}$$

Since the σ_i are not all of the same sign, by linearity there exists a tuple $\boldsymbol{\beta}$ such that

$$-\sigma_s(\sigma_1\beta_1 + \cdots + \sigma_{s-1}\beta_{s-1}) > 0,$$

with $0 < \beta_i \leq |\lambda_0|$. One can now assume that there exists a large positive number D that depends on β_i, such that $\frac{1}{D} \leq \beta_i \leq |\lambda_0|$. Hence, there exists an open neighbourhood of positive measure over which the integrand on the right hand side of (2.6.16) is positive. Therefore we deduce that $\Psi(0) \gg 1$.

The asymptotic formula (2.6.15) together with (2.6.6) yields,

$$\int_{\mathfrak{M}} f_1(\alpha) \cdots f_s(\alpha) K_{\pm}(\alpha) \mathrm{d}\alpha = 2\tau\Omega(s,\theta;\boldsymbol{\lambda})P^{s-\theta} + o\left(P^{s-\theta}\right).$$

The proof of Theorem 2.1.1 is now complete by taking into account Lemma 2.4.3, Lemma 2.5.1 and the expression (2.2.11).

2.7 The inhomogeneous case

In this section we prove Theorem 2.1.2. Using the kernel functions defined in (2.2.3) we have that

$$R_-(P) \leq \mathcal{N}_{s,\theta}^{\tau}(P;\boldsymbol{\lambda},L) \leq R_+(P),$$

whereas now
$$R_{\pm}(P) = \int_{-\infty}^{\infty} f_1(\alpha)\cdots f_s(\alpha)e(-\alpha L)K_{\pm}(\alpha)\mathrm{d}\alpha.$$

To study the above integrals we dissect the real line as in the case of Theorem 2.1.1. By the triangle inequality and appealing to Lemma 2.4.3 and Lemma 2.5.1, we immediately obtain that

$$\left|\int_{\mathfrak{m}\cup\mathfrak{t}} f_1(\alpha)\cdots f_s(\alpha)e(-\alpha L)K_{\pm}(\alpha)\mathrm{d}\alpha\right| \ll \int_{\mathfrak{m}\cup\mathfrak{t}} |f_1(\alpha)\cdots f_s(\alpha)|\,\mathrm{d}\alpha = o\left(P^{s-\theta}\right).$$

Thus, we are left to deal with the contribution arising when integrating over the major arc around zero. The approach given in §2.6 applies here as well with minor adjustments, in order to deal with the twist $e(-\alpha L)$. We briefly now discuss these differences.

The singular integral is now given by

$$\mathcal{I}_{\pm} = \int_{-\infty}^{\infty} v_1(\alpha)\cdots v_s(\alpha)e(-\alpha L)K_{\pm}(\alpha)\mathrm{d}\alpha,$$

where the functions $v_i(\alpha)$ are defined as in (2.6.1). One may show as in §2.6 that

$$\mathcal{I}_{\pm} = \int_{\mathfrak{M}} f_1(\alpha)\cdots f_s(\alpha)e(-\alpha L)K_{\pm}(\alpha)\mathrm{d}\alpha + o\left(P^{s-\theta}\right).$$

Thus, we aim to give an asymptotic formula for the complete singular integral $I_{\pm}$ defined above.

For $\alpha \in \mathbb{R}$ we now define

$$\Phi(\alpha) = v_1(\alpha)\cdots v_s(\alpha)e(-\alpha L) = \int_{[0,P]^s} e\left(\alpha\left(\lambda_1\gamma_1^\theta + \cdots + \lambda_s\gamma_s^\theta - L\right)\right)\mathrm{d}\boldsymbol{\gamma}. \tag{2.7.1}$$

Ignoring for the moment the twist factor $e(-\alpha L)$ one can study the function Φ as in §2.6. This analysis leads now to

$$\Phi(\alpha) = \left(\frac{P}{\theta}\right)^s |\lambda_1\cdots\lambda_s|^{-1/\theta}\int_{-\infty}^{\infty}\Psi(\widetilde{\beta})e\left(\alpha\left(P^\theta\sigma_s\widetilde{\beta} - L\right)\right)\mathrm{d}\widetilde{\beta},$$

where Ψ is defined as in (2.6.9). Applying now Fourier's inversion theorem we obtain that

$$\mathcal{I}_{\pm} = \left(\frac{P}{\theta}\right)^s |\lambda_1\cdots\lambda_s|^{-1/\theta}\int_{-\infty}^{\infty}\Psi(\widetilde{\beta})\left(\int_{-\infty}^{\infty} e\left(\alpha\left(P^\theta\sigma_s\widetilde{\beta} - L\right)\right)K_{\pm}(\alpha)\mathrm{d}\alpha\right)\mathrm{d}\widetilde{\beta}. \tag{2.7.2}$$

One may assume that $\widetilde{\beta}$ satisfies

$$\int_{-\infty}^{\infty} e\left(\alpha\left(P^\theta\sigma_s\widetilde{\beta} - L\right)\right)K_{\pm}(\alpha)\mathrm{d}\alpha = \begin{cases} 1, & \text{if } |\widetilde{\beta} - LP^{-\theta}| < \tau P^{-\theta}, \\ 0, & \text{otherwise.}\end{cases} \tag{2.7.3}$$

Note that for the measure of the set of points $\widetilde{\beta}$ which do not satisfy the above relation one has $O\left(\tau|L^{-1}|P^{-\theta}\left(\log P\right)^{-1}\right) = o\left(P^{s-\theta}\right)$. The contribution of such points to the integral is $o\left(P^{s-\theta}\right)$ and hence one may ignore this set. Under the assumption that $|\widetilde{\beta} - LP^{-\theta}| < \tau P^{-\theta}$ one can show that
$$\Psi(\widetilde{\beta}) - \Psi(1) \ll (\tau + |L^{-1}|)P^{-\theta}.$$

Indeed, since the factor $e(-\alpha L)$ in (2.7.1) does not affect things, one can repeat the argument given in the proof of Lemma 2.6.1 to deduce that

$$\Psi(\widetilde{\beta}) - \Psi(0) \ll |\widetilde{\beta}| \ll |\widetilde{\beta} - LP^{-\theta}| + |LP^{-\theta}| \ll (\tau + |L|)\, P^{-\theta}.$$

One can now substitute $\Psi(\widetilde{\beta}) = \Psi(0) + (\tau + |L|)\, P^{-\theta}$ into (2.7.2). In view of (2.7.3) this yields

$$\mathcal{I}_{\pm} = \left(\frac{P}{\theta}\right)^s |\lambda_1 \cdots \lambda_s|^{-1/\theta} \int_{-\tau P^{-\theta}+LP^{-\theta}}^{\tau P^{-\theta}+LP^{-\theta}} \left(\Psi(0) + (\tau + |L|)\, P^{-\theta}\right) \mathrm{d}\widetilde{\beta}.$$

Therefore, we deduce that

$$\mathcal{I}_{\pm} = 2\tau \Omega(s, \theta; \boldsymbol{\lambda}) P^{s-\theta} + O\left((\tau + |L|)\, P^{s-2\theta}\right),$$

where

$$\Omega(s, \theta; \boldsymbol{\lambda}) = \left(\frac{1}{\theta}\right)^s |\lambda_1 \cdots \lambda_s|^{-1/\theta} \Psi(0) > 0,$$

with $\Psi(0)$ given by (2.6.9). The proof of Theorem 2.1.2 is now complete.

2.8 The definite case

In this section we prove Theorem 2.1.3. Here we deal with positive definite generalised polynomials. In this section we put

$$\mathcal{F}(\mathbf{x}) = \lambda_1 x_1^\theta + \cdots + \lambda_s x_s^\theta - \nu,$$

and recall that we write $\rho_s(\tau, \nu)$ to denote the number of solutions $\mathbf{x} \in \mathbb{N}^s$ possessed by the inequality $|\mathcal{F}(\mathbf{x})| < \tau$ for a fixed real number $\tau \in (0, 1]$. Our approach follows that presented in [22, Theorem 1.10], where the authors deal with the problem of counting solutions to inequalities for positive definite polynomials.

Let N be a positive large parameter. For any solution $\mathbf{x}$ counted by $\rho_s(\tau, \nu)$ with $\nu \leq N$ one has $0 < x_i \leq P \ (1 \leq i \leq s)$, where

$$P = 2\left(\lambda_1^{-1/\theta} + \cdots + \lambda_s^{-1/\theta} + 1\right) N^{1/\theta}. \tag{2.8.1}$$

So, one can write

$$\rho_s(\tau, \nu) = \sum_{\substack{\mathbf{x} \in [1, P]^s \\ |\mathcal{F}(\mathbf{x})| < \tau}} 1.$$

Recall the kernel function $K(\alpha) = \mathrm{sinc}^2(\alpha)$. For any real $\eta > 0$ we define the function

$$w_\eta(x) = \eta K(\eta x)$$

that was used in [30]. It satisfies

$$w_\eta(x) \ll \min\{1, |x|^{-2}\} \quad \text{and} \quad 0 \leq w_\eta(x) \leq \eta. \tag{2.8.2}$$

The Fourier transform of this function is given by

$$\widehat{w}_\eta(x) = \int_{-\infty}^{\infty} w_\eta(u)e(-xu)\mathrm{d}u = \max\left\{0, 1 - \frac{|x|}{\eta}\right\}. \tag{2.8.3}$$

Now we define the weighted integral

$$\rho_s^\star(\tau,\nu) = \int_{-\infty}^{\infty} f_1(\alpha)\cdots f_s(\alpha)w_\tau(\alpha)\mathrm{d}\alpha.$$

In the light of the discussion in [22, §2.1, §2.2] and appealing to [22, Lemma 2.1], whenever $0 < \Delta < \frac{\tau}{2}$ one has

$$\rho_s(\tau,\nu) = \left(1 + \frac{\tau}{\Delta}\right)\rho_s^\star(\tau+\Delta,\nu) - \frac{\tau}{\Delta}\rho_s^\star(\tau,\nu) + O\left(\rho_s^\star(\Delta,\nu+\tau) + \rho_s^\star(\Delta,\nu-\tau)\right). \tag{2.8.4}$$

It is apparent by (2.8.4) that it is enough to establish an asymptotic formula for the weighted integral $\rho_s^\star(\tau,\nu)$. To do so, we dissect the real line into three disjoint sets as in §2.2. Note that now we take P as defined in (2.8.1).

For estimating the contribution arising from the sets of minor and trivial arcs one can invoke Lemma 2.4.3 and Lemma 2.5.1. Together with the fact that by (2.8.2) one has $w_\tau(\alpha) \ll 1$ for any α, we deduce that

$$\int_{\mathfrak{m}\cup\mathfrak{t}} |f_1(\alpha)\cdots f_s(\alpha)e(-\alpha\nu)w_\tau(\alpha)|\,\mathrm{d}\alpha = o\left(P^{s-\theta}\right). \tag{2.8.5}$$

So, one is left to deal with the contribution arising when integrating over the major arc. We write

$$I(\mathfrak{M}) = \int_{\mathfrak{M}} f_1(\alpha)\cdots f_s(\alpha)e(-\alpha\nu)w_\tau(\alpha)\mathrm{d}\alpha,$$

and the singular integral is given by

$$\mathcal{I}_\infty = \int_{-\infty}^{\infty} v_1(\alpha)\cdots v_s(\alpha)e(-\alpha\nu)w_\tau(\alpha)\mathrm{d}\alpha,$$

where the functions $v_i(\alpha)$ are defined as in (2.6.1). Below we obtain an asymptotic formula for the integral $I(\mathfrak{M})$. The argument is analogous to the one given in [22, Lemma 2.4].

Lemma 2.8.1. *Provided that $s > 2\theta$ one has*

$$I(\mathfrak{M}) = \frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)}(\lambda_1\cdots\lambda_s)^{-1/\theta}\tau\nu^{s/\theta-1} + O\left(\tau\left(P^{s-\theta-1+\delta_0} + P^{s-\theta-\delta_0(s/\theta-1)}\right)\right),$$

uniformly in $0 < \tau \le 1$ and $1 \le \nu \le N$.

Proof. Using the fact that $0 \le w_\tau(x) \le \tau$ one has as in (2.6.2) that

$$I(\mathfrak{M}) - \int_{\mathfrak{M}} v_1(\alpha)\cdots v_s(\alpha)e(-\alpha\nu)w_\tau(\alpha)\mathrm{d}\alpha \ll \tau P^{s-\theta-1+2\delta_0}.$$

So as in §2.6 we may infer that

$$\int_{\mathbb{R}\setminus\mathfrak{M}} v_1(\alpha)\cdots v_s(\alpha)e(-\alpha\nu)w_\tau(\alpha)\mathrm{d}\alpha \ll \tau \int_{|\alpha|>P^{-\theta+\delta_0}} |\alpha|^{-s/\theta}\mathrm{d}\alpha$$

$$\ll \tau P^{s-\theta-\delta_0(s/\theta-1)}.$$

Thus, the above two estimates yield

$$I(\mathfrak{M}) = \mathcal{I}_\infty + O\left(\tau\left(P^{s-\theta-1+\delta_0} + P^{s-\theta-\delta_0(s/\theta-1)}\right)\right). \tag{2.8.6}$$

By (2.6.1) we may write

$$\mathcal{I}_\infty = \int_{-\infty}^{\infty}\left(\int_{[0,P]^s} e(\alpha(\lambda_1\gamma_1^\theta+\cdots+\lambda_s\gamma_s^\theta))\right)e(-\alpha\nu)w_\tau(\alpha)\mathrm{d}\alpha.$$

Since the integral is absolutely convergent we can interchange the order of integration in the right hand side of the above formula. Invoking (2.8.3) one has

$$\mathcal{I}_\infty = \int_{[0,P]^s} \widehat{w}_\tau\left(\lambda_1\gamma_1^\theta+\cdots+\lambda_s\gamma_s^\theta-\nu\right)\mathrm{d}\boldsymbol{\gamma}.$$

Since $\lambda_i > 0$ one may use (2.8.3) to extend the order of integration to $[0,\infty)^s$. After a change of variables with $\gamma_i = \beta_i\lambda_i^{-1/\theta}$ $(1\le i\le s)$ the above expression takes the shape

$$\mathcal{I}_\infty = (\lambda_1\cdots\lambda_s)^{-1/\theta}\int_{[0,\infty)^s} \widehat{w}_\tau\left(\beta_1^\theta+\cdots+\beta_s^\theta-\nu\right)\mathrm{d}\boldsymbol{\beta}. \tag{2.8.7}$$

Consider the level sets of the function $\beta_1^\theta+\cdots+\beta_s^\theta$. For $t\in\mathbb{R}$ the equation $t=\beta_1^\theta+\cdots+\beta_s^\theta$ defines a surface in $\mathbb{R}^s$ of codimension 1. We write $\mathcal{S}$ to denote the surface obtained by the intersection with the domain $\{(\beta_1,\ldots,\beta_s):\beta_i>0\ (1\le i\le s)\}\subset\mathbb{R}^s$. The area of $\mathcal{S}$ is equal to

$$t^{s/\theta-1}\frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)}.$$

Using the transformation formula we may integrate over $\mathcal{S}$ and applying Fubini's theorem equation (2.8.7) takes the shape

$$\mathcal{I}_\infty = (\lambda_1\cdots\lambda_s)^{-1/\theta}\frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)}\int_0^\infty t^{s/\theta-1}\widehat{w}_\tau(t-\nu)\mathrm{d}t. \tag{2.8.8}$$

By (2.8.3) and putting $t-\nu=u$ one has

$$\int_0^\infty t^{s/\theta-1}\widehat{w}_\tau(t-\nu)\mathrm{d}t = \int_{-\tau}^{\tau}\left(1-\frac{|u|}{\tau}\right)(\nu+u)^{s/\theta-1}\mathrm{d}u.$$

For large enough ν one has $|u/\nu|<1$, so the binomial expansion yields

$$(\nu+u)^{s/\theta-1} = \nu^{s/\theta-1}\left(1+\frac{u}{\nu}\right)^{s/\theta-1} = \nu^{s/\theta-1}+O\left(u\nu^{s/\theta-2}\right),$$

as $\nu \to \infty$. By this asymptotic expansion one has

$$\int_{-\tau}^{\tau} \left(1 - \frac{|u|}{\tau}\right) (\nu + u)^{s/\theta-1} \mathrm{d}u = \tau\nu^{s/\theta-1} + O\left(\tau^2\nu^{s/\theta-2}\right).$$

Returning to (2.8.8) we deduce that

$$\mathcal{I}_\infty = (\lambda_1 \cdots \lambda_s)^{-1/\theta} \frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)} \tau\nu^{s/\theta-1} + O\left(\tau^2\nu^{s/\theta-2}\right),$$

which when combined with (2.8.6) and observing that $\tau P^{s-\theta-1-\delta_0} \gg \tau^2\nu^{s/\theta-2}$ completes the proof of the lemma. □

We may now complete the proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. Putting together (2.8.5) and the conclusion of Lemma 2.8.1 we deduce that

$$\rho_s^\star(\tau,\nu) = \frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)} (\lambda_1 \cdots \lambda_s)^{-1/\theta} \tau\nu^{s/\theta-1} + o\left(\nu^{s/\theta-1}\right).$$

One can now substitute the above formula into (2.8.4). This yields

$$\rho_s(\tau,\nu) = (\lambda_1 \cdots \lambda_s)^{-1/\theta} \frac{\Gamma\left(1+\frac{1}{\theta}\right)^s}{\Gamma\left(\frac{s}{\theta}\right)} \left(2\tau\nu^{s/\theta-1} + \Delta\nu^{s/\theta-1} + W\right) + o\left(\nu^{s/\theta-1}\right), \tag{2.8.9}$$

where

$$W = O\left(\Delta(\nu+\tau)^{s/\theta-1} + \Delta(\nu-\tau)^{s/\theta-1} + o\left((\nu+\tau)^{s/\theta-1}\right) + o\left((\nu-\tau)^{s/\theta-1}\right)\right).$$

Here Δ is at our disposal, as long as it satisfies $0 < \Delta < \frac{\tau}{2}$. One may choose $\Delta = \frac{\tau}{3}\nu^{-1/100}$. Then, as $\nu \to \infty$ one has

$$\Delta\nu^{s/\theta-1} + W = o\left(\nu^{s/\theta-1}\right).$$

Hence the asymptotic formula (2.8.9) delivers the desired conclusion which completes the proof. □

2.9 A discrete L^2-restriction estimate

This section is devoted to the demonstration of Theorem 2.1.5. Before we present our proof let us motivate the route we take. To make this clearer assume for the moment that $\theta = d \in \mathbb{N}$. Then an application of the Cauchy-Schwarz inequality reveals that

$$\int_0^1 \left| \sum_{1 \le x \le P} \mathfrak{a}_x e(\alpha x^d) \right|^{2s} \mathrm{d}\alpha = \int_0^1 \left| \sum_{\ell \in \mathbb{Z}} \sum_{\mathbf{x} \in \mathcal{B}_d(\ell)} \mathfrak{a}_{x_1} \cdots \mathfrak{a}_{x_s} e(\alpha\ell) \right|^2 \mathrm{d}\alpha$$
$$\le \sum_{\ell \in \mathbb{Z}} (\#\left(\mathcal{B}_d(\ell)\right)) \sum_{\mathbf{x} \in \mathcal{B}_d(\ell)} |\mathfrak{a}_{x_1} \cdots \mathfrak{a}_{x_s}|^2,$$

where $\mathcal{B}_d(\ell) = \{1 \leq \mathbf{x} \leq P : x_1^d + \cdots + x_s^d = \ell\}$, and we write $\#(\mathcal{B}_d(\ell))$ to denote its cardinality. By orthogonality one has

$$\#(\mathcal{B}_d(\ell)) = \int_0^1 \left| \sum_{1 \leq x \leq P} e(\alpha x^d) \right|^s e(-\alpha \ell) \mathrm{d}\alpha.$$

Hence the problem boils down to bounding the quantity $\max_\ell \#(\mathcal{B}_d(\ell))$. Using classical methods together with the circle method, one can show that $\#(\mathcal{B}_d(\ell)) \ll P^{s-d}$ for sufficiently large s. For example, using the latest method of Wooley [95] on Vinogradov's mean value theorem, one can take $s \geq s_0$ with s_0 as in (2.1.7).

When dealing with $\theta \notin \mathbb{N}$ one has to vary slightly from the argument sketched above. As an analogue of $\mathcal{B}_d(\ell)$ we define the set

$$\mathcal{B}_\theta(\ell) = \left\{1 \leq \mathbf{x} \leq P : |x_1^\theta + \cdots + x_s^\theta - \ell| < 1/2\right\}.$$

The partition

$$\bigcup_{\ell \in \mathbb{Z}} \mathcal{B}_\theta(\ell) = \{(x_1, \ldots, x_s) : 1 \leq x_i \leq P\}$$

no longer makes sense for a fractional exponent θ. In this situation we instead look at tuples $\mathbf{x}$ such that $x_1^\theta + \cdots + x_s^\theta$ is close to an integer value ℓ. This observation makes apparent the link between our aim and the problem of representing integers by a generalized polynomial as described in (2.1.6).

Note that with the notation of §2.8, one has $K(\alpha) = w_1(\alpha)$ and so by (2.8.3) one has

$$\int_{-\infty}^{\infty} e(\alpha \xi) K(\alpha) \mathrm{d}\alpha = \max\{0, 1 - |\xi|\}, \tag{2.9.1}$$

for all $\xi \in \mathbb{R}$. We may now embark to the proof.

Proof of Theorem 2.1.5. Recall that we assume $s \geq 2\left(\lfloor 2\theta \rfloor + 1\right)\left(\lfloor 2\theta \rfloor + 2\right) + 2$. Expanding one has that

$$\int_{-\infty}^{\infty} |f_{\mathfrak{a}}(\alpha)|^{2s} K(\alpha) \mathrm{d}\alpha = \int_{-\infty}^{\infty} \left| \sum_{1 \leq \mathbf{x} \leq P} \mathfrak{a}_{x_1} \cdots \mathfrak{a}_{x_s} e\left(\alpha(x_1^\theta + \cdots + x_s^\theta)\right) \right|^2 K(\alpha) \mathrm{d}\alpha. \tag{2.9.2}$$

By the definition of the nearest integer function $\|\cdot\|_{\mathbb{R}/\mathbb{Z}} : \mathbb{R} \to [0, 1/2]$ we can decompose the summation over $1 \leq \mathbf{x} \leq P$ by counting integer solutions of the inhomogeneous inequality

$$|x_1^\theta + \cdots + x_s^\theta - \ell_1| \leq 1/2,$$

inside the box $[1, P]^s$, where ℓ_1 runs over $\mathbb{Z}$. With this observation and expanding the square, one has that the right hand side of (2.9.2) is equal to

$$\sum_{\ell_1, \ell_2 \in \mathbb{Z}} \sum_{\mathbf{x} \in \mathcal{B}_\theta(\ell_1)} \sum_{\mathbf{y} \in \mathcal{B}_\theta(\ell_2)} \mathfrak{a}_{x_1} \cdots \mathfrak{a}_{x_s} \overline{\mathfrak{a}_{y_1}} \cdots \overline{\mathfrak{a}_{y_s}} \int_{-\infty}^{\infty} e\left(\alpha\left(\sigma_{s,\theta}(\mathbf{x}, \mathbf{y})\right)\right) K(\alpha) \mathrm{d}\alpha, \tag{2.9.3}$$

where we write $\sigma_{s,\theta}(\mathbf{x}, \mathbf{y}) = x_1^\theta + \cdots + x_s^\theta - y_1^\theta - \cdots - y_s^\theta$. We emphasize at this point that it is at this step where we essentially "double" the number of variables.

Invoking (2.9.1) we see that

$$0 < \int_{-\infty}^{\infty} e\left(\alpha(\sigma_{s,\theta}(\mathbf{x},\mathbf{y})\right) K(\alpha)\mathrm{d}\alpha \leq 1$$

if and only if $|\sigma_{s,\theta}(\mathbf{x},\mathbf{y})| < 1$. Indeed, if $|\sigma_{s,\theta}(\mathbf{x},\mathbf{y})| \geq 1$ then $\max\{0, 1 - |\sigma_{s,\theta}(\mathbf{x},\mathbf{y})|\} = 0$ and so by (2.9.1) we see that the expression in (2.9.3) is equal to zero. Hence, in this case there is nothing to prove since the estimate claimed in the statement of Theorem 2.1.5 trivially holds. Thus, we may assume that the tuples $\mathbf{x},\mathbf{y}$ satisfy the inequality $|\sigma_{s,\theta}(\mathbf{x},\mathbf{y})| < 1$. Under this assumption one has

$$\begin{aligned}\sum_{\ell_1,\ell_2\in\mathbb{Z}} \sum_{\mathbf{x}\in\mathcal{B}_\theta(\ell_1)} \sum_{\mathbf{y}\in\mathcal{B}_\theta(\ell_2)} \mathfrak{a}_{x_1}\cdots\mathfrak{a}_{x_s}\overline{\mathfrak{a}_{y_1}}\cdots\overline{\mathfrak{a}_{y_s}} \int_{-\infty}^{\infty} e\left(\alpha\left(\sigma_{s,\theta}(\mathbf{x},\mathbf{y})\right)\right) K(\alpha)\mathrm{d}\alpha \\ \ll \sum_{\ell_1,\ell_2\in\mathbb{Z}} \sum_{\mathbf{x}\in\mathcal{B}_\theta(\ell_1)} \sum_{\substack{\mathbf{y}\in\mathcal{B}_\theta(\ell_2) \\ |\sigma_{s,\theta}(\mathbf{x},\mathbf{y})|<1}} \mathfrak{a}_{x_1}\cdots\mathfrak{a}_{x_s}\overline{\mathfrak{a}_{y_1}}\cdots\overline{\mathfrak{a}_{y_s}}.\end{aligned} \tag{2.9.4}$$

Let $(\mathbf{x},\mathbf{y}) \in \mathcal{B}_\theta(\ell_1)\times\mathcal{B}_\theta(\ell_2)$ and suppose that $|\sigma_{s,\theta}(\mathbf{x},\mathbf{y})| < 1$. Then by the triangle inequality one has

$$|\ell_1 - \ell_2| \leq \left|\ell_1 - (x_1^\theta + \cdots + x_s^\theta)\right| + |\sigma_{s,\theta}(\mathbf{x},\mathbf{y})| + \left|\ell_2 - (y_1^\theta + \cdots + y_s^\theta)\right| < 2.$$

Therefore, it turns out that

$$\begin{aligned}\sum_{\ell_1,\ell_2\in\mathbb{Z}} \sum_{\mathbf{x}\in\mathcal{B}_\theta(\ell_1)} \sum_{\substack{\mathbf{y}\in\mathcal{B}_\theta(\ell_2) \\ |\sigma_{s,\theta}(\mathbf{x},\mathbf{y})|<1}} \mathfrak{a}_{x_1}\cdots\mathfrak{a}_{x_s}\overline{\mathfrak{a}_{y_1}}\cdots\overline{\mathfrak{a}_{y_s}} \\ \leq \sum_{\ell_1,\ell_2\in\mathbb{Z}} \sum_{\mathbf{x}\in\mathcal{B}_\theta(\ell_1)} \sum_{\substack{\mathbf{y}\in\mathcal{B}_\theta(\ell_2) \\ |\ell_1-\ell_2|<2}} \mathfrak{a}_{x_1}\cdots\mathfrak{a}_{x_s}\overline{\mathfrak{a}_{y_1}}\cdots\overline{\mathfrak{a}_{y_s}}.\end{aligned} \tag{2.9.5}$$

Since $\ell_1-\ell_2 \in \{0, \pm 1\}$ we see that if we fix one of the ℓ_1, ℓ_2 the other one has exactly 3 choices. So by symmetry one has that the expression on the right hand side of (2.9.5) is bounded above by

$$6\sum_{\ell_3\in\mathbb{Z}} \left|\sum_{\mathbf{x}\in\mathcal{B}'_\theta(\ell_3)} \mathfrak{a}_{x_1}\cdots\mathfrak{a}_{x_s}\right|^2,$$

where for $\ell_3 \in \mathbb{Z}$ we put

$$\mathcal{B}'_\theta(\ell_3) = \{1 \leq \mathbf{x} \leq P : |x_1^\theta + \cdots + x_s^\theta - \ell_3| < 1\}.$$

An application of the Cauchy-Schwarz inequality reveals that for $s \geq 2\left(\lfloor 2\theta\rfloor + 1\right)\left(\lfloor 2\theta\rfloor + 2\right) + 2$

one has

$$6 \sum_{\ell_3 \in \mathbb{Z}} \left| \sum_{\mathbf{x} \in \mathcal{B}'_\theta(\ell_3)} \mathfrak{a}_{x_1} \cdots \mathfrak{a}_{x_s} \right|^2 \leq 6 \sum_{\ell_3 \in \mathbb{Z}} \left(\sum_{\mathbf{x} \in \mathcal{B}'_\theta(\ell_3)} 1 \right) \left(\sum_{\mathbf{x} \in \mathcal{B}'_\theta(\ell_3)} |\mathfrak{a}_{x_1} \cdots \mathfrak{a}_{x_s}|^2 \right)$$

$$\ll \max_{\ell_3 \in \mathbb{Z}} (\#\mathcal{B}'_\theta(\ell_3)) \sum_{\ell_3 \in \mathbb{Z}} \sum_{\mathbf{x} \in \mathcal{B}'_\theta(\ell_3)} |\mathfrak{a}_{x_1} \cdots \mathfrak{a}_{x_s}|^2$$

$$\ll P^{s-\theta} \left(\sum_{1 \leq x \leq P} |\mathfrak{a}_x|^2 \right)^s,$$

where in the last step we used Theorem 2.1.3. Putting together (2.9.4), (2.9.5) and invoking (2.9.2) we are done. □

Chapter 3

A mixed Diophantine system

The work in this chapter is based (with minor changes) on the author's paper [65].

3.1 Introduction

In this chapter we investigate the simultaneous solubility of inequalities and equations. Here we seek to count the number of positive integer solutions of a mixed system, consisting of a diagonal inequality of fractional degree and a diagonal integral form.

Fix non-zero real numbers λ_i, μ_j not all of the same sign and non-zero integers a_i, b_k not all of the same sign. Suppose that $d \geq 2$ is an integer and suppose further that $\theta > d+1$ is real and non-integral. We write

$$\begin{cases} \mathfrak{F}(\mathbf{x}, \mathbf{y}) = \lambda_1 x_1^\theta + \cdots + \lambda_\ell x_\ell^\theta + \mu_1 y_1^\theta + \cdots + \mu_m y_m^\theta \\ \\ \mathfrak{D}(\mathbf{x}, \mathbf{z}) = a_1 x_1^d + \cdots a_\ell x_\ell^d + b_1 z_1^d + \cdots + b_n z_n^d. \end{cases} \tag{3.1.1}$$

Let τ be a fixed positive real number. The Diophantine system under investigation is of the shape

$$\begin{cases} |\mathfrak{F}(\mathbf{x}, \mathbf{y})| < \tau \\ \\ \mathfrak{D}(\mathbf{x}, \mathbf{z}) = 0. \end{cases} \tag{3.1.2}$$

In order to ensure that the system (3.1.2) is indefinite it is enough to ask for the system

$$\mathfrak{F}(\mathbf{x}, \mathbf{y}) = \mathfrak{D}(\mathbf{x}, \mathbf{z}) = 0, \tag{3.1.3}$$

to admit a non-trivial real solution $(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Beyond the indefiniteness of $\mathfrak{F}$ and $\mathfrak{D}$, in order to study the solubility of the system (3.1.2) over the set of natural numbers one has to impose some further conditions. It is apparent that we must ask for the congruence $\mathfrak{D}(\mathbf{x}, \mathbf{z}) \equiv 0 \pmod{p^\nu}$ to be soluble for all prime powers p^ν. Furthermore, for reasons associated with the application of the circle method, one has to assume that the given local solutions are in fact non-singular. For us a tuple $\boldsymbol{\eta} = (\mathbf{x}^\star, \mathbf{y}^\star, \mathbf{z}^\star) \in \mathbb{R}^s$ which satisfies the system of equations (3.1.3) is called a non-singular solution of the system (3.1.2) if the Jacobian matrix

$$\frac{\partial(\mathfrak{F},\mathfrak{D})}{\partial(\eta_1,\ldots,\eta_s)}$$

has full rank. We say that the system (3.1.2) satisfies the *local solubility condition* if the system (3.1.3) possesses a non-singular real solution and the congruence $\mathfrak{D}(\mathbf{x},\mathbf{z}) \equiv 0 \pmod{p^\nu}$ possesses a non-singular solution for all prime powers p^ν.

We write $\mathcal{N}(P)$ to denote the number of positive integer solutions $(\mathbf{x},\mathbf{y},\mathbf{z})$ of the system (3.1.2) with

$$\frac{1}{2}\mathbf{x}^\star P < \mathbf{x} \leq 2\mathbf{x}^\star P, \quad \frac{1}{2}\mathbf{y}^\star P < \mathbf{y} \leq 2\mathbf{y}^\star P, \quad \frac{1}{2}\mathbf{z}^\star P < \mathbf{z} \leq 2\mathbf{z}^\star P,$$

where $(\mathbf{x}^\star,\mathbf{y}^\star,\mathbf{z}^\star)$ is a non-singular solution of the system (3.1.3). Our aim is to establish an asymptotic formula for the counting function $\mathcal{N}(P)$ as $P \to \infty$. Before we state our result we make a comment about two special cases. Suppose that $\ell = 0$. It is apparent by Theorem 2.1.1 and the seminal work of Davenport and Lewis [31] that in such a case and provided that $m \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)+1$ and $n \geq d^2+1$, one certainly has $\mathcal{N}(P) \gg P^{m+n-(\theta+d)}$. Suppose now that $m = n = 0$. Here one would (in principle) be able to obtain an asymptotic formula for the counting function $\mathcal{N}(P)$ provided that $s = \ell \geq \ell_0(\theta)+1$, where $\ell(\theta)$ is any natural number for which one has the estimate

$$\int_0^1\int_0^1 \left|\sum_{1\leq x\leq P} e(\alpha_d x^d + \alpha_\theta x^\theta)\right|^{\ell_0(\theta)} \mathrm{d}\boldsymbol{\alpha} \ll P^{\ell_0(\theta)-\theta+\epsilon}.$$

Here $\mathrm{d}\boldsymbol{\alpha}$ stands for $\mathrm{d}\alpha_d \mathrm{d}\alpha_\theta$. Our first result is establishing this observation.

Theorem 3.1.1. *Suppose that $d \geq 2$ is an integer and suppose further that $\theta > d+1$ is real and non-integral. Let τ be a fixed positive real number. Consider the system*

$$|\mathfrak{F}(\mathbf{x},\mathbf{y})| < \tau \quad \textit{and} \quad \mathfrak{D}(\mathbf{x},\mathbf{z}) = 0, \tag{3.1.4}$$

with $\mathfrak{F},\mathfrak{D}$ defined in (3.1.1). Suppose that $m = n = 0$ and suppose further that the system (3.1.4) satisfies the local solubility condition, namely the system (3.1.3) possesses a non-singular real solution and the congruence $\mathfrak{D}(\mathbf{x},\mathbf{z}) \equiv 0 \pmod{p^\nu}$ possesses a non-singular solution for all prime powers p^ν. Then, provided that $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)+1$, one has that there exists a positive real number $C = C(\boldsymbol{\lambda},\boldsymbol{a},\theta,d,s)$ such that

$$\mathcal{N}(P) = 2\tau C P^{s-(\theta+d)} + o\left(P^{s-(\theta+d)}\right), \tag{3.1.5}$$

as $P \to \infty$. In particular, the number of positive integer solutions $\mathbf{x} \in [1,P]^s$ of the system (3.1.4) is $\gg P^{s-(\theta+d)}$, where the implicit constant is a positive real number, which depends on $s, \lambda_i, a_i, \theta, d$ and τ.

Certainly more interesting is the case where in (3.1.1) one has $m+n \neq 0$. Our next result examines this case when the total number of variables s is in an intermediate range compared to the number of variables needed in the scenarios where $\ell = 0$ and $m = n = 0$.

Theorem 3.1.2. *Suppose that $d \geq 2$ is an integer and suppose further that $\theta > d+1$ is real and non-integral. Let τ be a fixed positive real number. Consider the system*

$$|\mathfrak{F}(\mathbf{x}, \mathbf{y})| < \tau \quad \textit{and} \quad \mathfrak{D}(\mathbf{x}, \mathbf{z}) = 0, \tag{3.1.6}$$

with $\mathfrak{F}, \mathfrak{D}$ defined in (3.1.1). We write

$$A_\theta = (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) \quad \textit{and} \quad A_d = d^2. \tag{3.1.7}$$

Moreover we set

$$s_{\min} = \left\lceil \max\left\{ A_\theta + n,\ \frac{A_d}{A_\theta} m + A_\theta \right\} \right\rceil + 1$$

and

$$s_{\max} = \left\lfloor \min\left\{ A_\theta + A_d,\ A_\theta + \frac{A_d}{A_\theta} m + n \right\} \right\rfloor + 1.$$

Suppose that the system (3.1.6) satisfies the following conditions.

(a) The system (3.1.6) satisfies the local solubility condition, namely the system (3.1.3) possesses a non-singular real solution and the congruence $\mathfrak{D}(\mathbf{x}, \mathbf{z}) \equiv 0 \pmod{p^\nu}$ possesses a non-singular solution for all prime powers p^ν.

(b) One has $\ell \geq \max\{\lceil 2\theta(1 - n/d) \rceil,\ 1\},\ 0 \leq m \leq A_\theta$ and $0 \leq n \leq A_d$.

(c) One has $\ell + m \geq A_\theta + 1$ and $\ell + n \geq A_d + 1$.

(d) For the total number of variables $s = \ell + m + n$ one has $s_{\min} \leq s \leq s_{\max}$.

Then, there exists a positive real number $C = C(\boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{a}, \mathbf{b}, \theta, d, s)$, such that as $P \to \infty$ one has

$$\mathcal{N}(P) = 2\tau C P^{s-(\theta+d)} + o\left(P^{s-(\theta+d)}\right). \tag{3.1.8}$$

In particular, the number of positive integer solutions $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in [1, P]^\ell \times [1, P]^m \times [1, P]^n$ of the system (3.1.6) is $\gg P^{s-(\theta+d)}$, where the implicit constant is a positive real number, which depends on $s, \lambda_i, \mu_j, a_i, b_k, \theta, d$ and τ.

The positive real number $C = C(\boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{a}, \mathbf{b}, \theta, d, s)$ appearing in the asymptotic formulae (3.1.5) and (3.1.8) turns out to be a product of the shape $C = \mathfrak{J}_0 \mathfrak{S}$. Here

$$\mathfrak{J}_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{\mathcal{B}} e\left(\beta_\theta \mathfrak{F}(\mathbf{x}, \mathbf{y}) + \beta_d \mathfrak{D}(\mathbf{x}, \mathbf{z})\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} \right) \mathrm{d}\boldsymbol{\beta},$$

where

$$\mathcal{B} = \bigtimes_{i=1}^{\ell} \left[\frac{1}{2} x_i^\star, 2x_i^\star \right] \bigtimes_{j=1}^{m} \left[\frac{1}{2} y_j^\star, 2y_j^\star \right] \bigtimes_{k=1}^{n} \left[\frac{1}{2} z_k^\star, 2z_k^\star \right]$$

is a box containing in its interior a non-singular solution $(\mathbf{x}^\star, \mathbf{y}^\star, \mathbf{z}^\star)$ of the system (3.1.3), and

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} T(q, a),$$

where

$$T(q,a) = q^{-(\ell+n)} \prod_{i=1}^{\ell} S(q, aa_i) \prod_{k=1}^{n} S(q, ab_k),$$

and for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ we write

$$S(q,a) = \sum_{z=1}^{q} e\left(\frac{az^d}{q}\right).$$

The singular integral $\mathfrak{J}_0$ is essentially Schmidt's singular integral. The singular series $\mathfrak{S}$ captures the arithmetic behind the equation $\mathfrak{D}(\mathbf{x}, \mathbf{z}) = 0$.

By the assumptions made in Theorem 3.1.2 we see that our conclusion is valid for systems for which the total number of variables $s = \ell + m + n$ satisfies $A_\theta + 1 \le s \le A_\theta + A_d + 1$. Note that when $m = n = 0$ in Theorem 3.1.1 we assume that $s = \ell \ge A_\theta + 1$, with A_θ defined in (3.1.7). The treatment of the minor arcs in the proof of Theorem 3.1.1 follows by using a Hua's type inequality

$$\int_{\mathcal{B}} |f(\alpha_d, \alpha_\theta)|^s \,\mathrm{d}\boldsymbol{\alpha} \ll \left(\sup_{(\alpha_d, \alpha_\theta) \in \mathcal{B}} |f(\alpha_d, \alpha_\theta)| \right)^{s-2t} \int_{\mathcal{B}} |f(\alpha_d, \alpha_\theta)|^{2t} \,\mathrm{d}\boldsymbol{\alpha},$$

as in Chapter 2, where for $(\alpha_d, \alpha_\theta) \in \mathbb{R}^2$ we write

$$f(\alpha_d, \alpha_\theta) = \sum_{1 \le x \le P} e(\alpha_d x^d + \alpha_\theta x^\theta),$$

and where $\mathcal{B}$ is a Lebesgue measurable subset of $\mathbb{R}^2$. For the case where $m + n \neq 0$ one may adopt the methods we use in proving Theorem 3.1.2 to treat systems where the total number of variables is greater than $A_\theta + A_d + 1$. For such cases one may obtain the following corollary.

Corollary 3.1.3. *Suppose that $d \ge 2$ is an integer and suppose further that $\theta > d + 1$ is real and non-integral. Let τ be a fixed positive real number. Consider the system*

$$|\mathfrak{F}(\mathbf{x}, \mathbf{y})| < \tau \quad \textit{and} \quad \mathfrak{D}(\mathbf{x}, \mathbf{z}) = 0, \tag{3.1.9}$$

with $\mathfrak{F}, \mathfrak{D}$ defined in (3.1.1). Suppose that the system (3.1.9) satisfies the following conditions.

(a) The system satisfies the local solubility condition, namely the system (3.1.3) possesses a non-singular real solution and the congruence $\mathfrak{D}(\mathbf{x}, \mathbf{z}) \equiv 0 \pmod{p^\nu}$ possesses a non-singular solution for all prime powers p^ν.

(b) One has $\ell \ge \max\{\lceil 2\theta(1 - n/d) \rceil,\ 1\},\ 0 \le m \le A_\theta$ and $0 \le n \le A_d$, with A_θ and A_d as in (3.1.7).

(c) One has $\ell + m \ge A_\theta + 1$ and $\ell + n \ge A_d + 1$, with A_θ and A_d as in (3.1.7).

(d) One has $s = \ell + m + n \ge A_\theta + A_d + 2$.

Then, the number of positive integer solutions $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in [1, P]^\ell \times [1, P]^m \times [1, P]^n$ of the system (3.1.9) is $\gg P^{s-(\theta+d)}$, where the implicit constant is a positive real number, which depends on $s, \lambda_i, \mu_j, a_i, b_k, \theta, d$ and τ.

Having stated our results let us make a few comments regarding previous works that are of some relevance to the problem we investigate. The study of Diophantine inequalities for diagonal real forms begins with the work of Davenport and Heilbronn [30]. Many authors have engaged with studying the solubility of systems of diagonal real forms of the same degree. The interested reader may look for example in [18], [19], [26], [62], [51]. For the case of unlike degrees we have the important work of Schmidt [67] who studied systems of real (not necessarily diagonal) forms of differing odd degrees. In this work Schmidt proves the existence (without being explicitly determined) of a finite lower bound for the number of variables needed to ensure solubility. For the first time, an explicit such bound was given by Freeman [42] in the case of a system of cubic forms.

Using ideas from [4], Freeman in [36] and [39] introduced a variant of the Davenport—Heilbronn method. These results of Freeman were afterwards improved by Wooley in [86] using an amplification method. Based on his variant of the original Davenport–Heilbronn method, Freeman considered systems of diagonal quadratic real forms in [38] and systems of diagonal real forms of degree d in [41]. A two dimensional analogue of the Davenport–Heilbronn method was presented by Parsell in [56]. Shortly afterwards, in [57] and [59], Parsell adapted Freeman's method to study the solubility of systems of diagonal real forms of unlike degree. More recently, we have Chow's paper [25] which is an inequality analogue of Birch's celebrated result [6]. The interested reader may look as well in the recent breakthroughs due to Myerson [49] and [50], who obtained a remarkable improvement compared to Birch's theorem for systems of quadratic and cubic integral forms.

3.2 Set up

3.2.1 An analytic representation for the counting function $\mathcal{N}(P)$

Set $\widetilde{\tau} = \tau(\log P)^{-1}$. We put

$$K_{\pm}(\alpha) = \frac{\sin(\pi\alpha\widetilde{\tau})\sin(\pi\alpha(2\tau \pm \widetilde{\tau}))}{\pi^2\alpha^2\widetilde{\tau}}. \tag{3.2.1}$$

By [39, Lemma 1] and its proof we know that

$$K_{\pm}(\alpha) \ll_{\tau} \min\{1, |\alpha|^{-1}, (\log P)|\alpha|^{-2}\}, \tag{3.2.2}$$

and

$$0 \leq \int_{-\infty}^{\infty} e(\xi\alpha)K_{-}(\alpha)\mathrm{d}\alpha \leq \chi_{\tau}(\xi) \leq \int_{-\infty}^{\infty} e(\xi\alpha)K_{+}(\alpha)\mathrm{d}\alpha \leq 1, \tag{3.2.3}$$

where we write $\chi_{\tau}(\xi)$ to denote the indicator function of the interval $(-\tau, \tau)$, namely

$$\chi_{\tau}(\xi) = \begin{cases} 1, & \text{if } \ |\xi| < \tau, \\ 0, & \text{if } \ |\xi| \geq \tau. \end{cases}$$

Note that the expression

$$\left|\int_{-\infty}^{\infty} e(\xi\alpha)K_{\pm}(\alpha)\mathrm{d}\alpha - \chi_{\tau}(\xi)\right|$$

is zero when $||\xi| - \tau| > \widetilde{\tau}$ and at most 1 for values of ξ such that $||\xi| - \tau| \le \widetilde{\tau}$.

One may rewrite the kernel functions $K_\pm(\alpha)$ defined in (3.2.1) in the shape

$$K_\pm(\alpha) = (2\tau \pm \widetilde{\tau}) \frac{\sin(\pi\alpha\widetilde{\tau})}{\pi\alpha\widetilde{\tau}} \cdot \frac{\sin(\pi\alpha(2\tau \pm \widetilde{\tau}))}{\pi\alpha(2\tau \pm \widetilde{\tau})}.$$

Using a Taylor expansion one has for $|x| < 1$ with $x \neq 0$ that

$$\frac{\sin x}{x} = 1 + O(x^2).$$

Recall that $\widetilde{\tau} = \tau(\log P)^{-1}$. So for $|\alpha| < 1$ and P sufficiently large one has that

$$K_\pm(\alpha) = 2\tau + O\left((\log P)^{-2}\right). \tag{3.2.4}$$

In our analysis we use various exponential sums. For $\boldsymbol{\alpha} = (\alpha_d, \alpha_\theta) \in \mathbb{R}^2$ we define the exponential sums $f(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}; P),\ g(\alpha_\theta) = g(\alpha_\theta; P)$ and $h(\alpha_d) = h(\alpha_d; P)$ by

$$f(\boldsymbol{\alpha}; P) = \sum_{1\le x\le P} e(\alpha_d x^d + \alpha_\theta x^\theta), \qquad g(\alpha_\theta; P) = \sum_{1\le x\le P} e(\alpha_\theta x^\theta),$$

$$h(\alpha_d; P) = \sum_{1\le x\le P} e(\alpha_d x^d).$$

Moreover, we define $F_i(\boldsymbol{\alpha}) = F_i(\boldsymbol{\alpha}; P),\ G_j(\alpha_\theta) = G_j(\alpha_\theta; P)$ and $H_k(\alpha_d) = H(\alpha_d; P)$ by

$$F_i(\boldsymbol{\alpha}; P) = \sum_{1\le x\le P} e(a_i\alpha_d x^d + \lambda_i\alpha_\theta x^d) \qquad (1 \le i \le \ell),$$

$$G_j(\alpha_\theta; P) = \sum_{1\le x\le P} e(\mu_j\alpha_\theta x^\theta) \qquad (1 \le j \le m),$$

$$H_k(\alpha_d; P) = \sum_{1\le x\le P} e(b_k\alpha_d x^d) \qquad (1 \le k \le n).$$

Recall that $(\mathbf{x}^\star, \mathbf{y}^\star, \mathbf{z}^\star)$ is a non-singular real solution of the system (3.1.3). We put

$$f_i(\boldsymbol{\alpha}) = \sum_{\frac{1}{2}x_i^\star P < x \le 2x_i^\star P} e(a_i\alpha_d x^d + \lambda_i\alpha_\theta x^\theta) \qquad (1 \le i \le \ell),$$

$$g_j(\alpha_\theta) = \sum_{\frac{1}{2}y_j^\star P < y \le 2y_j^\star P} e(\mu_j\alpha_\theta y^\theta) \qquad (1 \le j \le m),$$

$$h_k(\alpha_d) = \sum_{\frac{1}{2}z_k^\star P < z \le 2z_k^\star P} e(b_k\alpha_d z^d) \qquad (1 \le k \le n).$$

For future reference we note here the following relations

$$f_i(\boldsymbol{\alpha}) = F\left(\boldsymbol{\alpha}; 2x_i^\star P\right) - F\left(\boldsymbol{\alpha}; \frac{1}{2}x_i^\star P\right), \quad g_j(\alpha_\theta) = G\left(\alpha_\theta; 2y_j^\star P\right) - G\left(\alpha_\theta; \frac{1}{2}y_j^\star P\right),$$
$$h_k(\alpha_d) = H\left(\alpha_d; 2z_k^\star P\right) - H\left(\alpha_d; \frac{1}{2}z_k^\star P\right). \tag{3.2.5}$$

We define the generating function

$$\mathcal{F}(\boldsymbol{\alpha}) = \prod_{i=1}^{\ell} f_i(\boldsymbol{\alpha}) \prod_{j=1}^{m} g_j(\alpha_\theta) \prod_{k=1}^{n} h_k(\alpha_d),$$

and set

$$R_\pm(P) = \int_{-\infty}^{\infty} \int_0^1 \mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta) \mathrm{d}\boldsymbol{\alpha}. \tag{3.2.6}$$

Using now (3.2.3), together with the usual orthogonality relation

$$\int_0^1 e(\alpha n) \mathrm{d}\alpha = \begin{cases} 1, & \text{when } n = 0, \\ \\ 0, & \text{when } n \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

one has that

$$R_-(P) \le \mathcal{N}(P) \le R_+(P).$$

From the above inequality it is clear that in order to establish an asymptotic formula for the counting function $\mathcal{N}(P)$ it suffices to obtain asymptotic formulae for the integrals $R_\pm(P)$ that are asymptotically equal.

3.2.2 A mixed version of the circle method

In order to study the integrals $R_\pm(P)$ defined in (3.2.6) we apply a mixed version of the circle method. We dissect separately $\mathbb{R}$ and $[0, 1)$.

Dissection of $\mathbb{R}$. Here we apply a Davenport–Heilbronn dissection. Write $\gamma = \theta - \lfloor \theta \rfloor \in (0, 1)$ for the fractional part of θ. Define the parameters $\delta_0 = \delta_0(\theta)$ and $\omega = \omega(\theta)$ by

$$\delta_0(\theta) = 2^{1-2\theta} \qquad \text{and} \qquad \omega(\theta) = \min\left\{\frac{1-\gamma}{12}, \frac{1}{5^{100(\theta+d)}}\right\}. \tag{3.2.7}$$

Define the set of major, minor, and trivial arcs respectively as follows

$$\mathfrak{M} = \left\{\alpha_\theta \in \mathbb{R} : |\alpha_\theta| < P^{-\theta+\delta_0}\right\},$$

$$\mathfrak{m} = \left\{\alpha_\theta \in \mathbb{R} : P^{-\theta+\delta_0} \le |\alpha_\theta| < P^{\omega}\right\},$$

$$\mathfrak{t} = \left\{\alpha_\theta \in \mathbb{R} : |\alpha_\theta| \ge P^{\omega}\right\}.$$

Dissection of $[0, 1)$. Here we apply a classical Hardy–Littlewood dissection into major and minor arcs. Pick a parameter ξ satisfying

$$0 < \xi \leq \frac{\delta_0}{8}. \tag{3.2.8}$$

For integers a, q such that $0 \leq a < q \leq P^{\xi}$ and $(a, q) = 1$, we define a major arc around the rational fraction a/q to be the set

$$\mathfrak{N}_{\xi}(q, a) = \{\alpha_d \in [0, 1) : \ |\alpha_d - a/q| < P^{-d+\xi}\}.$$

We now form the union

$$\mathfrak{N}_{\xi} = \bigcup_{\substack{0 \leq a < q \leq P^{\xi} \\ (a,q)=1}} \mathfrak{N}_{\xi}(q, a),$$

and call this the set of major arcs. Note that $\mathfrak{N}_{\xi}$ is a union of disjoint sets. Indeed, suppose that there exists $\alpha_d \in [0, 1)$ which belongs to two distinct major arcs $\mathfrak{N}_{\xi}(q_1, a_1), \mathfrak{N}_{\xi}(q_2, a_2) \subset \mathfrak{N}_{\xi}$. Since $a_1/q_1 \neq a_2/q_2$ one has

$$\frac{1}{q_1 q_2} \leq \left| \frac{a_1 q_2 - a_2 q_1}{q_1 q_2} \right| \leq 2P^{-d+\xi},$$

which in turn implies that $1 \leq 2q_1 q_2 P^{-d+\xi} \leq 2P^{-d+3\xi}$. This is clearly impossible for large P, since by our choice in (3.2.8) one has $\xi < 1/3$. The set of minor arcs is defined to be the complement of the set of major arcs. Denote this set by $\mathfrak{n}_{\xi}$. Namely we have

$$\mathfrak{n}_{\xi} = [0, 1) \setminus \mathfrak{N}_{\xi}.$$

Using the above dissections one may express $[0, 1) \times \mathbb{R}$ into a disjoint union of sets of the shape

$$[0, 1) \times \mathbb{R} = \mathfrak{P} \cup \mathfrak{p} \cup \mathfrak{c},$$

where we define the sets $\mathfrak{P}$, $\mathfrak{p}$ and $\mathfrak{c}$ as follows.

(1) The set of major arcs $\mathfrak{P}$ given by

$$\mathfrak{P} = \mathfrak{N}_{\xi} \times \mathfrak{M}.$$

(2) The set of minor arcs $\mathfrak{p}$ given by

$$\mathfrak{p} = ([0, 1) \times \mathfrak{m})) \cup (\mathfrak{n}_{\xi} \times \mathfrak{M}).$$

(3) The set of trivial arcs $\mathfrak{c}$ given by

$$\mathfrak{c} = [0, 1) \times \mathfrak{t}.$$

For a Lebesgue measurable $\mathcal{B} \subset [0, 1) \times \mathbb{R}$ we define

$$R_{\pm}(P; \mathcal{B}) = \int_{\mathcal{B}} \mathcal{F}(\boldsymbol{\alpha}) K_{\pm}(\alpha_{\theta}) \mathrm{d}\boldsymbol{\alpha}. \tag{3.2.9}$$

Recalling (3.2.6), one has that

$$R_{\pm}(P) = R_{\pm}(P; \mathfrak{P}) + R_{\pm}(P; \mathfrak{p}) + R_{\pm}(P; \mathfrak{c}). \tag{3.2.10}$$

3.2.3 An application of Hölder's inequality

We begin by recalling the well known inequality

$$|z_1 \cdots z_n| \ll |z_1|^n + \cdots |z_n|^n,$$

which is valid for all complex numbers z_i. Let $\mathcal{B}$ be a Lebesgue measurable set. An application of this inequality reveals that for some indices i, j and k one has

$$|\mathcal{F}(\boldsymbol{\alpha})| \ll f_i^\ell g_j^m h_k^n,$$

where for easy of notation we abbreviate

$$|f_i(\alpha_d, \alpha_\theta)| \text{ to } f_i, \quad |g_j(\alpha_\theta)| \text{ to } \; g_j \text{ and } \; |h_k(\alpha_d)| \text{ to } h_k.$$

Let $\delta \in [0, 1/3)$ be a real number at our disposal to be chosen at a later stage. We write

$$\ell' = \ell - \delta \quad \text{and} \quad s' = \ell' + m + n = s - \delta. \tag{3.2.11}$$

Note here that $\ell', s' \notin \mathbb{N}$. The previous estimate yields

$$\int_{\mathcal{B}} |\mathcal{F}(\boldsymbol{\alpha}) K_{\pm}(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha} \ll \left(\sup_{\mathcal{B}} |f_i| \right)^{\delta} \int_{\mathcal{B}} f_i^{\ell'} g_j^m h_k^n |K_{\pm}(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha}, \tag{3.2.12}$$

where the supremum is taken over $\boldsymbol{\alpha} = (\alpha_\theta, \alpha_d) \in \mathcal{B}$.

We define the following auxiliary mean values,

$$\Xi_{f_i}(\mathcal{B}) = \int_{\mathcal{B}} f_i^{A_\theta} |K_{\pm}(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha}, \qquad \Xi_{f_i, g_j}(\mathcal{B}) = \int_{\mathcal{B}} f_i^{A_d} g_j^{A_\theta} |K_{\pm}(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha},$$

$$\Xi_{f_i, h_k}(\mathcal{B}) = \int_{\mathcal{B}} f_i^{A_\theta} h_k^{A_d} |K_{\pm}(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha}, \qquad \Xi_{g_j, h_k}(\mathcal{B}) = \int_{\mathcal{B}} g_j^{A_\theta} h_k^{A_d} |K_{\pm}(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha}.$$

For $\omega_i \in (0, 1)$ a formal application of Hölder's inequality reveals

$$\begin{aligned} &\int_{\mathcal{B}} f_i^{\ell'} g_j^m h_k^n |K_{\pm}(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha} \\ &\quad \ll \left(\Xi_{f_i}(\mathcal{B})\right)^{\omega_1} \left(\Xi_{f_i, g_j}(\mathcal{B})\right)^{\omega_2} \left(\Xi_{f_i, h_k}(\mathcal{B})\right)^{\omega_3} \left(\Xi_{g_j, h_k}(\mathcal{B})\right)^{\omega_4}. \end{aligned} \tag{3.2.13}$$

Combining (3.2.13) and (3.2.12) yields

$$\begin{aligned}&\int_{\mathcal{B}}|\mathcal{F}(\boldsymbol{\alpha})K_{\pm}(\alpha_\theta)|\mathrm{d}\boldsymbol{\alpha}\\ &\qquad \ll \left(\sup_{\mathcal{B}}|f_i|\right)^{\delta}\left(\Xi_{f_i}(\mathcal{B})\right)^{\omega_1}\left(\Xi_{f_i,g_j}(\mathcal{B})\right)^{\omega_2}\left(\Xi_{f_i,h_k}(\mathcal{B})\right)^{\omega_3}\left(\Xi_{g_j,h_k}(\mathcal{B})\right)^{\omega_4}.\end{aligned} \tag{3.2.14}$$

The task now is to prove that there exist admissible values ω_i such that the inequality (3.2.13) is valid. The $\omega_i \in (0,1)$ must satisfy the simultaneous linear equations

$$\begin{cases} A_\theta\omega_1 + A_d\omega_2 + A_\theta\omega_3 = \ell' \\ A_\theta\omega_2 + A_\theta\omega_4 = m \\ A_d\omega_3 + A_d\omega_4 = n \\ \omega_1+\omega_2+\omega_3+\omega_4 = 1. \end{cases}$$

By the two equations in the middle we infer that

$$\omega_2 = \omega_3 + \frac{m}{A_\theta} - \frac{n}{A_d}.$$

Substituting $\omega_2+\omega_4 = m/A_\theta$ into the last equation of the system yields

$$\omega_1 = -\omega_3 + 1 - \frac{m}{A_\theta}.$$

One may substitute into the first equation of the system the above values for ω_2 and ω_1. Hence

$$\omega_3 = \frac{s'-A_\theta}{A_d} - \frac{m}{A_\theta}.$$

Having determined a value for ω_3 one may solve for ω_1, ω_2 and ω_4 to obtain

$$\omega_1 = 1 - \frac{s'-A_\theta}{A_d}, \qquad \omega_2 = \frac{s'-A_\theta}{A_d} - \frac{n}{A_d}, \qquad \omega_4 = \frac{m}{A_\theta} + \frac{n}{A_d} - \frac{s'-A_\theta}{A_d}. \tag{3.2.15}$$

We now have to ensure that $\omega_i \in (0,1)$. Since $\omega_1+\omega_2+\omega_3+\omega_4 = 1$ it suffices to ensure that $\omega_i > 0$. Solving the simultaneous inequalities $\omega_i > 0 \ (1 \le i \le 4)$ yields

$$\max\left\{A_\theta + n,\ \frac{A_d}{A_\theta}m + A_\theta\right\} \le s' \le \min\left\{A_\theta + A_d,\ A_\theta + \frac{A_d}{A_\theta}m + n\right\}.$$

Note that this is a legitimate constrain since we assume that $0 \le m \le A_\theta$ and $0 \le n \le A_d$.

Next, we deduce a constraint for s. Recall from (3.2.11) that $s' = s - \delta$. Since we consider s to be a natural number, the preceding inequality about the range of s' now delivers

$$\left\lceil \delta + \max\left\{A_\theta + n,\ \frac{A_d}{A_\theta}m + A_\theta\right\}\right\rceil \le s \le \left\lfloor \delta + \min\left\{A_\theta + A_d,\ A_\theta + \frac{A_d}{A_\theta}m + n\right\}\right\rfloor.$$

For any $x, y \in \mathbb{R}$ one has

$$\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$$

$$\lceil x \rceil + \lceil y \rceil - 1 \le \lceil x + y \rceil \le \lceil x \rceil + \lceil y \rceil + 1.$$

Since $0 \le \delta < 1/3 < 1$ one has

$$\left\lfloor \delta + \min\left\{A_\theta + A_d,\ A_\theta + \frac{A_d}{A_\theta} m + n\right\}\right\rfloor \ge \left\lfloor \min\left\{A_\theta + A_d,\ A_\theta + \frac{A_d}{A_\theta} m + n\right\}\right\rfloor + 1,$$

and

$$\left\lceil \delta + \max\left\{A_\theta + n,\ \frac{A_d}{A_\theta} m + A_\theta\right\}\right\rceil \le \left\lceil \max\left\{A_\theta + n,\ \frac{A_d}{A_\theta} m + A_\theta\right\}\right\rceil + 1.$$

Hence one has

$$\left\lceil \max\left\{A_\theta + n,\ \frac{A_d}{A_\theta} m + A_\theta\right\}\right\rceil + 1 \le s \le \left\lfloor \min\left\{A_\theta + A_d,\ A_\theta + \frac{A_d}{A_\theta} m + n\right\}\right\rfloor + 1,$$

which is precisely the range prescribed by the condition (d) in the statement of Theorem 3.1.2. It is therefore clear that for such s the inequality (3.2.13) is valid.

3.3 Auxiliary mean value estimates

The aim of this section is to collect the necessary auxiliary estimates that we employ in the following sections. From now on, and for ease of notation, for each $j \in \{1, \dots, n, \theta\}$ we put

$$\sigma_{t,j}(\mathbf{x}) = \sum_{i=1}^{t} (x_i^j - x_{t+i}^j). \tag{3.3.1}$$

Lemma 3.3.1. *Suppose that $I \subset (0, \infty)$ is a bounded interval. Let δ be a given positive real number and define the number Δ by the relation $2\delta\Delta = 1$. We write $V_t(I; \delta)$ to denote the number of positive integer solutions $x_i \in I$ of the inequality*

$$|\sigma_{t,\theta}(\mathbf{x})| < \delta.$$

Then one has

$$\delta \int_{-\Delta}^{\Delta} \left| \sum_{x \in I} e(\alpha x^\theta) \right|^{2t} \mathrm{d}\alpha \ll V_t(I; \delta) \ll \delta \int_{-\Delta}^{\Delta} \left| \sum_{x \in I} e(\alpha x^\theta) \right|^{2t} \mathrm{d}\alpha,$$

with the implicit constants in the above estimate being independent from I, θ, and δ.

Proof. This is a special case of [81, Lemma 2.1] with $K = 1$ and $\phi(x) = x^\theta$ in their notation. Alternatively, this follows by Lemma 2.3.2 with $\mathcal{S} = I \times I$ and intervals $I_1 = I_3 = I$. □

Next, we need a variant of the above lemma that allows one to bound from above the mixed mean values $\Xi_{f_i, g_j}(\mathcal{B})$ and $\Xi_{f_i, h_k}(\mathcal{B})$, by the number of solutions of the corresponding underlying system. We write $Z_1(P)$ to denote the number of integer solutions of the system

$$\begin{cases} \left|\lambda_i \sigma_{\frac{A_d}{2},\theta}(\mathbf{x}) + \mu_j \sigma_{\frac{A_\theta}{2},\theta}(\mathbf{y})\right| < \dfrac{1}{2\kappa} \\ \\ a_i \sigma_{\frac{A_d}{2},d}(\mathbf{x}) = 0, \end{cases}$$

with $\frac{1}{2}x_i^\star P < \mathbf{x} \leq 2x_i^\star P$ and $\frac{1}{2}y_i^\star P < \mathbf{y} \leq 2y_i^\star P$. Similarly, we write $Z_2(P)$ to denote the number of integer solutions of the system

$$\begin{cases} \left|\lambda_i \sigma_{\frac{A_\theta}{2},\theta}(\mathbf{x})\right| < \dfrac{1}{2\kappa} \\ \\ a_i \sigma_{\frac{A_\theta}{2},d}(\mathbf{x}) + b_k \sigma_{\frac{A_d}{2},d}(\mathbf{z}) = 0, \end{cases}$$

with $\frac{1}{2}x_i^\star P < \mathbf{x} \leq 2x_i^\star P$ and $\frac{1}{2}z_i^\star P < \mathbf{z} \leq 2z_i^\star P$.

Lemma 3.3.2. *Let κ be a positive real number and write $\mathcal{B} = [-1,1] \times [-\kappa,\kappa]$. Then for each index i, j and k one has*

(i) $\Xi_{f_i,g_j}(\mathcal{B}) \ll \kappa Z_1(P)$ *;*

(ii) $\Xi_{f_i,h_k}(\mathcal{B}) \ll \kappa Z_2(P)$.

The implicit constants do not depend on κ.

Proof. We give the proof only of estimate (i). One can establish estimate (ii) in a similar fashion.

Fix indices i and j. For ease of notation we put

$$p(\mathbf{x},\mathbf{y}) = \lambda_i \sigma_{\frac{A_d}{2},\theta}(\mathbf{x}) + \mu_j \sigma_{\frac{A_\theta}{2},\theta}(\mathbf{y}) \qquad \text{and} \qquad q(\mathbf{x}) = a_i \sigma_{\frac{A_d}{2},d}(\mathbf{x}).$$

Then, $Z_1(P)$ is equivalently given by the number of integer solutions of the system

$$\begin{cases} |p(\mathbf{x},\mathbf{y})| < \dfrac{1}{2\kappa} \\ |q(\mathbf{x})| < \dfrac{1}{2} \end{cases}$$

with $\frac{1}{2}x_i^\star P < \mathbf{x} \leq 2x_i^\star P$ and $\frac{1}{2}y_i^\star P < \mathbf{y} \leq 2y_i^\star P$.

Define the function

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & \text{when} \quad x \neq 0, \\ \\ 1, & \text{when} \quad x = 0. \end{cases}$$

By [30] we know that for each $x, \xi \in \mathbb{R}$ one has

$$\Lambda(x) = \int_{-\infty}^{\infty} e(x\xi)\operatorname{sinc}^2(\xi)\mathrm{d}\xi,$$

where for $x \in \mathbb{R}$ we write $\Lambda(x) = \max\{0, 1-|x|\}$. Note that one has $0 \leq \Lambda(x) \leq 1$. So for each solution counted by $Z_1(P)$ one has $0 < \Lambda(2\kappa p(\mathbf{x},\mathbf{y})) < 1$ and $0 < \Lambda(2q(\mathbf{x})) < 1$.

By the above considerations and taking the sum over the tuples $\mathbf{x}, \mathbf{y}$ with $\frac{1}{2}x_i^\star P < \mathbf{x} \leq 2x_i^\star P$ and $\frac{1}{2}y_i^\star P < \mathbf{y} \leq 2y_i^\star P$, we infer that

$$\begin{aligned}Z_1(P) &\geq \sum_{\mathbf{x},\mathbf{y}} \Lambda(2\kappa p(\mathbf{x},\mathbf{y}))\Lambda(2q(\mathbf{x}))\\ &= \sum_{\mathbf{x},\mathbf{y}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e\left(u_1\kappa p(\mathbf{x},\mathbf{y}) + u_2 2q(\mathbf{x})\right) \operatorname{sinc}^2(u_1)\operatorname{sinc}^2(u_2)\mathrm{d}\mathbf{u}\\ &= \frac{1}{4\kappa}\sum_{\mathbf{x},\mathbf{y}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e\left(\alpha_\theta p(\mathbf{x},\mathbf{y}) + \alpha_d q(\mathbf{x})\right) \operatorname{sinc}^2\left(\frac{1}{2\kappa}\alpha_\theta\right)\operatorname{sinc}^2\left(\frac{1}{2}\alpha_d\right)\mathrm{d}\boldsymbol{\alpha},\end{aligned}$$

where in the last step we applied a change of variables under the transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\kappa} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \alpha_\theta \\ \alpha_d \end{pmatrix}.$$

Because we have a finite sum and since the integral is absolutely convergent, one can change the order. Thus by the above inequality we obtain

$$Z_1(P) \geq \frac{1}{4\kappa}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_i^{A_d} g_j^{A_\theta} \operatorname{sinc}^2\left(\frac{1}{2\kappa}\alpha_\theta\right)\operatorname{sinc}^2\left(\frac{1}{2}\alpha_d\right)\mathrm{d}\boldsymbol{\alpha}. \tag{3.3.2}$$

Next, we use Jordan's inequality, which states that for $0 < x \leq \frac{\pi}{2}$ one has

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1.$$

For a proof of this inequality see [48, p. 33]. One then has $\operatorname{sinc}^2(x) > 4/\pi^2$ for $|x| < \frac{1}{2}$. Thus, for $|\alpha_\theta| < \kappa$ and $|\alpha_d| < 1$ one has

$$\operatorname{sinc}^2\left(\frac{1}{2\kappa}\alpha_\theta\right), \quad \operatorname{sinc}^2\left(\frac{1}{2}\alpha_d\right) > 4/\pi^2.$$

Hence, the inequality (3.3.2) now delivers

$$Z_1(P) \gg \frac{1}{\kappa}\int_{-\kappa}^{\kappa}\int_{-1}^{1} f_i^{A_d} g_j^{A_\theta}\mathrm{d}\boldsymbol{\alpha},$$

which completes the proof. □

For a tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n, \alpha_\theta) \in \mathbb{R}^{n+1}$ we put $T(\boldsymbol{\alpha}) = T(\boldsymbol{\alpha}; P)$, where

$$T(\boldsymbol{\alpha}; P) = \sum_{1 \leq x \leq P} e(\alpha_1 x + \cdots + \alpha_n x^n + \alpha_\theta x^\theta). \tag{3.3.3}$$

We need the following general estimate.

Theorem 3.3.3. *Let $\kappa \geq 1$ be a real number and suppose that $t \geq A_\theta/2$ is a natural number. Then, for any fixed $\epsilon > 0$ one has*

$$\int_{-\kappa}^{\kappa}\int_{[0,1)^n} |T(\boldsymbol{\alpha})|^{2t}\,\mathrm{d}\boldsymbol{\alpha} \ll \kappa P^{2t - \frac{1}{2}n(n+1) - \theta + \epsilon}.$$

The implicit constant depends on ϵ, θ, and t, but not on κ and P. Furthermore, for $t > A_\theta/2$ one can take $\epsilon = 0$.

Proof. This is Theorem B.1 in Appendix B. □

Next, we obtain essentially optimal mean value estimate for the exponential sums f, g and h.

Lemma 3.3.4. *Let $\kappa \geq 1$ be a real number. Then the following are valid.*

(i) Suppose that $t \geq A_\theta/2$ is a natural number. Then, for any fixed $\epsilon > 0$ one has

$$\int_{-\kappa}^{\kappa} \int_0^1 |f(\boldsymbol{\alpha})|^{2t} \, \mathrm{d}\boldsymbol{\alpha} \ll \kappa P^{2t-(\theta+d)+\epsilon}.$$

The implicit constant depends on ϵ, θ and t, but not on κ and P. Furthermore, for $t > A_\theta/2$ one can take $\epsilon = 0$.

(ii) Suppose that $t \geq A_\theta/2$ is a natural number. Then, for any fixed $\epsilon > 0$ one has

$$\int_{-\kappa}^{\kappa} |g(\alpha)|^{2t} \, \mathrm{d}\alpha \ll \kappa P^{2t-\theta+\epsilon}.$$

The implicit constant depends on ϵ, θ and t, but not on κ and P. Furthermore, for $t > A_\theta/2$ one can take $\epsilon = 0$.

(iii) Suppose that $t \geq A_d/2$ is a natural number. Then, for any fixed $\epsilon > 0$ one has

$$\int_0^1 |h(\alpha)|^{2t} \, \mathrm{d}\alpha \ll P^{2t-d+\epsilon}.$$

The implicit constant depends on ϵ, d and t, but not on P. Furthermore, for $t > A_d/2$ one can take $\epsilon = 0$.

Proof. We begin with the estimate in (iii). This follows from [95, Corollary 14.7] since $A_d \geq s_0(d)$, where $s_0(d)$ is defined as

$$s_0(d) = d(d-1) + \min_{0 \leq m < d} \frac{2d + m(m-1)}{m+1}.$$

The estimate in (ii) is Theorem 2.1.4. Alternative, one can apply an argument similar to the one present below for proving (i).

We now come to the estimate in (i). Temporarily we write $n = \lfloor \theta \rfloor$. Keep in mind that we suppose that $\theta > d+1$, and so one has $d < n$. In order to prove the estimate in (i) we apply an average process as in [87, Theorem 2.1]. For each $1 \leq j \leq n$ with $j \neq d$ and for a tuple $\mathbf{h} = (h_1, \ldots, h_{d-1}, h_{d+1}, \ldots, h_n) \in \mathbb{Z}^{n-1}$ we put

$$\delta(\mathbf{x}, \mathbf{h}) = \prod_{\substack{j=1 \\ j \neq d}}^{n} \int_0^1 e\left(\beta_j \left(\sigma_{t,j}(\mathbf{x}) - h_j\right)\right) \mathrm{d}\beta_j,$$

where recall from (3.3.1) the definition of $\sigma_{t,j}(\mathbf{x})$. Let us rewrite the exponential sum $T(\boldsymbol{\alpha})$ defined in (3.3.3) as

$$T(\boldsymbol{\beta},\alpha_d,\alpha_\theta)=\sum_{1\leq x\leq P}e(\beta_1x+\cdots+\beta_{d-1}x^{d-1}+\alpha_dx^d+\beta_{d+1}x^{d+1}+\cdots+\alpha_\theta x^\theta).$$

Note that

$$\begin{aligned}\int_{-\kappa}^{\kappa}\int_{[0,1)^n}|T(\boldsymbol{\beta},\alpha_d,\alpha_\theta)|^{2t}\,e\left(-\sum_{\substack{j=1\\ j\neq d}}^{n}\beta_jh_j\right)\mathrm{d}\boldsymbol{\beta}=\\ =\sum_{1\leq\mathbf{x}\leq P}\delta(\mathbf{x},\mathbf{h})\int_{-\kappa}^{\kappa}\int_0^1 e\left(\alpha_d\sigma_{t,d}(\mathbf{x})+\alpha_\theta\sigma_{t,\theta}(\mathbf{x})\right)\mathrm{d}\alpha_d\mathrm{d}\alpha_\theta.\end{aligned}\tag{3.3.4}$$

By orthogonality one has

$$\int_0^1 e\left(\beta_j\left(\sigma_{t,j}(\mathbf{x})-h_j\right)\right)\mathrm{d}\beta_j=\begin{cases}1, & \text{when }\ \sigma_{t,j}(\mathbf{x})=h_j,\\ 0, & \text{when }\ \sigma_{t,j}(\mathbf{x})\neq h_j.\end{cases}$$

It is apparent that for each fixed choice of $1\leq\mathbf{x}\leq P$ there is precisely one possible value for the tuple $\mathbf{h}\in\mathbb{Z}^{n-1}$. Moreover, for each j and for $1\leq\mathbf{x}\leq P$ one has $|\sigma_{t,j}(\mathbf{x})|\leq tP^j$. Hence

$$\sum_{|h_1|\leq tP}\cdots\sum_{|h_{d-1}|\leq tP^{d-1}}\sum_{|h_{d+1}|\leq tP^{d+1}}\cdots\sum_{|h_n|\leq tP^n}\delta(\mathbf{x},\mathbf{h})=1.\tag{3.3.5}$$

One may return to (3.3.4) and sum over tuples $\mathbf{h}$ satisfying $|h_j|\leq tP^j$ for each $1\leq j\leq n$ with $j\neq d$. Thus we obtain

$$\begin{aligned}\sum_{\mathbf{h}}\int_{-\kappa}^{\kappa}\int_{[0,1)^n}|T(\boldsymbol{\beta},\alpha_d,\alpha_\theta)|^{2t}\,e\left(-\sum_{\substack{j=1\\ j\neq d}}^{n}\beta_jh_j\right)\mathrm{d}\boldsymbol{\beta}=\\ =\sum_{1\leq\mathbf{x}\leq P}\left(\sum_{\mathbf{h}}\delta(\mathbf{x},\mathbf{h})\right)\int_{-\kappa}^{\kappa}\int_0^1 e\left(\alpha_d\sigma_{t,d}(\mathbf{x})+\alpha_\theta\sigma_{t,\theta}(\mathbf{x})\right)\mathrm{d}\alpha_d\mathrm{d}\alpha_\theta.\end{aligned}$$

Applying the triangle inequality and taking into account (3.3.5) one has

$$\begin{aligned}P^{\frac{1}{2}n(n+1)-d}\int_{-\kappa}^{\kappa}\int_{[0,1)^n}|T(\boldsymbol{\beta},\alpha_d,\alpha_\theta)|^{2t}\,\mathrm{d}\boldsymbol{\beta}\geq\\ \geq\sum_{1\leq\mathbf{x}\leq P}\int_{-\kappa}^{\kappa}\int_0^1 e\left(\alpha_d\sigma_{t,d}(\mathbf{x})+\alpha_\theta\sigma_{t,\theta}(\mathbf{x})\right)\mathrm{d}\alpha_d\mathrm{d}\alpha_\theta.\end{aligned}$$

Note now that

$$\sum_{1\leq\mathbf{x}\leq P}\int_{-\kappa}^{\kappa}\int_0^1 e\left(\alpha_d\sigma_{t,d}(\mathbf{x})+\alpha_\theta\sigma_{t,\theta}(\mathbf{x})\right)\mathrm{d}\alpha_d\mathrm{d}\alpha_\theta=\int_{-\kappa}^{\kappa}\int_0^1|f(\boldsymbol{\alpha})|^{2t}\,\mathrm{d}\boldsymbol{\alpha}.$$

Invoking Theorem 3.3.3, we deduce that for any fixed $\epsilon > 0$ one has

$$\int_{-\kappa}^{\kappa}\int_{0}^{1}|f(\boldsymbol{\alpha})|^{2t}\,\mathrm{d}\boldsymbol{\alpha} \ll P^{\frac{1}{2}n(n+1)-d}\cdot P^{2t-\frac{1}{2}n(n+1)-\theta+\epsilon}$$

$$\ll P^{2t-(\theta+d)+\epsilon},$$

which completes the proof. □

Below we obtain mean value estimates for the exponential sums f_i, g_j and h_k.

Lemma 3.3.5. *For each index i, j and k the following are valid.*

(i) Suppose that κ is a real number such that $\kappa|\lambda_i| \geq 1$. Suppose further that $t \geq A_\theta/2$ is a natural number. Then, for any fixed $\epsilon > 0$ one has

$$\int_{-\kappa}^{\kappa}\int_{0}^{1}|f_i(\boldsymbol{\alpha})|^{2t}\ \mathrm{d}\boldsymbol{\alpha} \ll \kappa P^{2t-(\theta+d)+\epsilon}.$$

The implicit constant depend on $\epsilon, \theta, t, \lambda_i$ and a_i, but not on κ and P. Furthermore, for $t > A_\theta/2$ one can take $\epsilon = 0$.

(ii) Suppose that κ is a real number such that $\kappa|\mu_j| \geq 1$. Suppose further that $t \geq A_\theta/2$ is a natural number. Then, for any fixed $\epsilon > 0$ one has

$$\int_{-\kappa}^{\kappa}|g_j(\alpha_\theta)|^{2t}\ \mathrm{d}\alpha_\theta \ll \kappa P^{2t-\theta+\epsilon}.$$

The implicit constant depend on ϵ, θ, t and μ_j, but not on κ and P. Furthermore, for $t > A_\theta/2$ one can take $\epsilon = 0$.

(iii) Suppose that $t \geq A_d/2$ is a natural number. Then, for any fixed $\epsilon > 0$, one has

$$\int_{0}^{1}|h_k(\alpha_d)|^{2t}\ \mathrm{d}\alpha_d \ll P^{2t-d+\epsilon}.$$

The implicit constant depends on ϵ, d, t and b_k, but not on P. Furthermore, for $t > A_d/2$ one can take $\epsilon = 0$.

Proof. We give a proof only for the estimate in (i). One may argue in a similar fashion to establish the estimates in (ii) and (iii).

Fix an index i. Recalling (3.2.5) we see that it suffices to prove the following estimate

$$\int_{-\kappa}^{\kappa}\int_{0}^{1}|F_i(\boldsymbol{\alpha})|^{2t}\,\mathrm{d}\boldsymbol{\alpha} \ll \kappa P^{2t-(\theta+d)+\epsilon}.$$

Making a change of variables by

$$\begin{pmatrix}\alpha_\theta\\ \alpha_d\end{pmatrix} = \begin{pmatrix}\frac{1}{|\lambda_i|} & 0\\ 0 & \frac{1}{|a_i|}\end{pmatrix}\begin{pmatrix}\beta_\theta\\ \beta_d\end{pmatrix},$$

yields
$$\int_{-\kappa}^{\kappa}\int_0^1 |F_i(\boldsymbol{\alpha})|^{2t}\,\mathrm{d}\boldsymbol{\alpha} = \frac{1}{|\lambda_i a_i|}\int_{-\kappa|\lambda_i|}^{\kappa|\lambda_i|}\int_0^{|a_i|} |f(\pm\beta_d,\pm\beta_\theta)|^{2t}\,\mathrm{d}\boldsymbol{\beta}.$$

One can chop the interval $[0,|a_i|]$ into at most $\lfloor |a_i| \rfloor + 1$ intervals of length at most one. Moreover, because of the 1-periodicity with respect to β_d one has

$$\begin{aligned}\int_{-\kappa|\lambda_i|}^{\kappa|\lambda_i|}\int_0^{|a_i|} |f(\pm\beta_d,\pm\beta_\theta)|^{2t}\,\mathrm{d}\boldsymbol{\beta} &\ll \sum_{n=0}^{\lfloor |a_i| \rfloor}\int_{-\kappa|\lambda_i|}^{\kappa|\lambda_i|}\int_n^{n+1} |f(\pm\beta_d,\pm\beta_\theta)|^{2t}\,\mathrm{d}\boldsymbol{\beta} \\ &\ll \int_{-\kappa|\lambda_i|}^{\kappa|\lambda_i|}\int_0^{1} |f(\pm\beta_d,\pm\beta_\theta)|^{2t}\,\mathrm{d}\boldsymbol{\beta}.\end{aligned}$$

Finally, if necessary, one can make one more change of variables. This together with the fact that $f(-\boldsymbol{\beta}) = \overline{f(\boldsymbol{\beta})}$ yields

$$\int_{-\kappa|\lambda_i|}^{\kappa|\lambda_i|}\int_0^{1} |f(\pm\beta_d,\pm\beta_\theta)|^{2t}\,\mathrm{d}\boldsymbol{\beta} = \int_{-\kappa|\lambda_i|}^{\kappa|\lambda_i|}\int_0^{1} |f(\beta_d,\beta_\theta)|^{2t}\,\mathrm{d}\boldsymbol{\beta}.$$

The conclusion now follows by applying Lemma 3.3.4. □

We now estimate the auxiliary mean values $\Xi_{f_i}, \Xi_{f_i,g_j}, \Xi_{f_i,h_k}$, and Ξ_{g_j,h_k}.

Lemma 3.3.6. *Let κ be a real number such that for each index i and j one has $\kappa|\lambda_i| \geq 1$ and $\kappa|\mu_j| \geq 1$. Let $\mathcal{B} = [0,1]\times[-\kappa,\kappa]$. Then, for each index i, j and k, and for any fixed $\epsilon > 0$ one has*

(i) $\Xi_{f_i}(\mathcal{B}) \ll \kappa P^{A_\theta-(\theta+d)+\epsilon}$;

(ii) $\Xi_{f_i,g_j}(\mathcal{B}) \ll \kappa P^{A_\theta+A_d-(\theta+d)+\epsilon}$;

(iii) $\Xi_{f_i,h_k}(\mathcal{B}) \ll \kappa P^{A_\theta+A_d-(\theta+d)+\epsilon}$;

(iv) $\Xi_{g_j,h_k}(\mathcal{B}) \ll \kappa P^{A_\theta+A_d-(\theta+d)+\epsilon}$.

The implicit constants in the above estimates may depend on $\theta, d, \lambda_i, \mu_j, a_i, b_k$ and ϵ, but not on κ and P.

Proof. In the following we make use of the fact that by (3.2.2) one has $|K_\pm(\alpha_\theta)| \ll 1$. The estimate (i) follows by part (i) of Lemma 3.3.5 with $t = A_\theta/2$. The proof of the estimate (iv) is straightforward. One may write

$$\Xi_{g_j,h_k} \ll \left(\int_{-\kappa}^{\kappa} |g_j(\alpha_\theta)|^{A_\theta}\,\mathrm{d}\alpha_\theta\right)\left(\int_0^1 |h_k(\alpha_d)|^{A_d}\,\mathrm{d}\alpha_d\right),$$

and the conclusion now follows by using (ii) and (iii) of Lemma 3.3.5.

Now we turn our attention to the estimate in (ii). Fix indices i and j. By Lemma 3.3.2 and since we assume that $\kappa \geq \max_{i,j}\left\{|\lambda_i|^{-1}, |\mu_j|^{-1}\right\}$ one has

$$\Xi_{f_i,g_j} \ll \kappa Z_1(P), \tag{3.3.6}$$

whereas now $Z_1(P)$ denotes the number of integer solutions of the system

$$\begin{cases} \left| \lambda_i \sum_{i=1}^{\frac{A_d}{2}} \left(x_i^\theta - x_{\frac{A_d}{2}+i}^\theta \right) + \mu_j \sum_{i=1}^{\frac{A_\theta}{2}} \left(y_i^\theta - y_{\frac{A_\theta}{2}+i}^\theta \right) \right| < M \\ a_i \sum_{i=1}^{\frac{A_d}{2}} \left(x_i^d - x_{\frac{A_d}{2}+i}^d \right) = 0, \end{cases} \tag{3.3.7}$$

with $\frac{1}{2}x_i^\star P < \mathbf{x} \leq 2x_i^\star P$ and $\frac{1}{2}y_i^\star P < \mathbf{y} \leq 2y_i^\star P$, where $M = \left(2 \max_{i,j} \left\{ |\lambda_i|^{-1}, |\mu_j|^{-1} \right\}\right)^{-1}$. By orthogonality, the number of integer solutions of the equation in (3.3.7) is counted by the mean value

$$\int_0^1 \left| \sum_{\frac{1}{2}x_i^\star P < x \leq 2x_i^\star P} e(\alpha x^d) \right|^{A_d} \mathrm{d}\alpha_d.$$

Note that

$$\left| \sum_{\frac{1}{2}x_i^\star P < x \leq 2x_i^\star P} e(\alpha x^d) \right| \ll |h(\alpha; 2x_i^\star P)| + \left| h\left(\alpha; \frac{1}{2}x_i^\star P\right) \right|.$$

So by Lemma 3.3.4 one has for any fixed $\epsilon > 0$ that

$$\int_0^1 \left| \sum_{\frac{1}{2}x_i^\star P < x \leq 2x_i^\star P} e(\alpha x^d) \right|^{A_d} \mathrm{d}\alpha_d \ll P^{A_d - d + \epsilon}.$$

Let us fix an integer solution $\mathbf{x}$ for the equation in (3.3.7). As we proved, this can be done by choosing among $O\left(P^{A_d-d+\epsilon}\right)$ possibilities. Substitute now these values into the inequality in (3.3.7). Then the first block of variables is fixed and so one has to count the number of solutions of the inhomogeneous inequality

$$\left| \mu_j \sum_{i=1}^{\frac{A_\theta}{2}} \left(y_i^\theta - y_{\frac{A_\theta}{2}+i}^\theta \right) + L \right| < M$$

with $\frac{1}{2}y_i^\star P < \mathbf{y} \leq 2y_i^\star P$, where $L = L(\lambda_i, \theta, d, \epsilon, \mathbf{x})$ is a fixed real number, determined by the choice we made for the tuple $\mathbf{x}$. We write $V_{A_\theta}^{(1)}(P)$ to denote the number of integer solutions of this inequality. As a consequence of Theorem 2.1.2 one has

$$V_{A_\theta}^{(1)}(P) \ll P^{A_\theta - \theta + \epsilon}.$$

Hence, we have showed that $Z_1(P) \ll P^{A_\theta + A_d - (\theta + d) + \epsilon}$ and in view of (3.3.6) the proof of (ii) is now complete.

Similarly we argue for (iii). Fix indices i and k. By Lemma 3.3.2 one has

$$\Xi_{f_i, h_k} \ll \kappa Z_2(P), \tag{3.3.8}$$

whereas now $Z_2(P)$ denotes the number of integer solutions of the system

$$\begin{cases} \left| \lambda_i \sum_{i=1}^{\frac{A_\theta}{2}} \left(x_i^\theta - x_{\frac{A_\theta}{2}+i}^\theta \right) \right| < M \\ a_i \sum_{i=1}^{\frac{A_\theta}{2}} \left(x_i^d - x_{\frac{A_\theta}{2}+i}^d \right) + b_k \sum_{i=1}^{\frac{A_d}{2}} \left(z_i^d - z_{\frac{A_d}{2}+i}^d \right) = 0, \end{cases} \tag{3.3.9}$$

with $\frac{1}{2}x_i^\star P < \mathbf{x} \leq 2x_i^\star P$ and $\frac{1}{2}z_i^\star P < \mathbf{z} \leq 2z_i^\star P$. We write $V_{A_\theta}^{(2)}(P)$ to denote the number of integer solutions of the inequality in (3.3.9). By Lemma 3.3.1 one has

$$V_{A_\theta}^{(2)}(P) \ll \int_{\frac{-M|\lambda_i|}{2}}^{\frac{M|\lambda_i|}{2}} \left| \sum_{\frac{1}{2}y_i^\star P < y \leq 2y_i^\star P} e(\alpha x^\theta) \right|^{A_\theta} \mathrm{d}\alpha.$$

As in (ii) we can show that for any fixed $\epsilon > 0$ one has

$$V_{A_\theta}^{(2)}(P) \ll P^{A_\theta - \theta + \epsilon}.$$

Fix a solution $\mathbf{x}$ counted by $V_{A_\theta}^{(2)}(P)$. Substitute these values into the equation of system in (3.3.9). Then the first block of variables becomes a fixed integer, say $C = C(\lambda_i, \theta, \epsilon, \mathbf{x})$, which depends on the choice we made for the tuple $\mathbf{x}$. Hence, this equation takes the shape

$$a_i C + b_k \sum_{i=1}^{\frac{A_d}{2}} \left(z_i^d - z_{\frac{A_d}{2}+i}^d \right) = 0.$$

Note that if b_k does not divide the product $a_i C$, then the above equation is not soluble in integers. In such a case $Z_2(P) = 0$ and the claimed estimate holds trivially. Hence, assuming that $b_k \mid (a_i C)$ we can rewrite it as

$$\sum_{i=1}^{\frac{A_d}{2}} \left(z_i^d - z_{\frac{A_d}{2}+i}^d \right) = C',$$

where $C' = C'(\lambda_i, a_i, b_k, \theta, \epsilon, \mathbf{x})$ is a fixed integer determined by the choice we made for the tuple $\mathbf{x}$. The number of integer solutions of this last equation is bounded above by the mean value

$$\int_0^1 \left| \sum_{\frac{1}{2}z_i^\star P < z \leq 2z_i^\star P} e(\alpha z^d) \right|^{A_d} e(-\alpha C') \mathrm{d}\alpha.$$

Again note that

$$\left| \sum_{\frac{1}{2}z_i^\star P < z \leq 2z_i^\star P} e(\alpha z^d) \right| \ll \left| h\left(\alpha; 2z_i^\star P\right) \right| + \left| h\left(\alpha; \frac{1}{2}z_i^\star P\right) \right|.$$

So, by the triangle inequality and invoking Lemma 3.3.4 we deduce that

$$\int_0^1 \left| \sum_{\frac{1}{2}z_i^\star P < z \leq 2z_i^\star P} e(\alpha z^d) \right|^{A_d} e(-\alpha C')\mathrm{d}\alpha \ll P^{A_d - d + \epsilon}.$$

Hence, we deduce that $Z_2(P) \ll P^{A_\theta + A_d(\theta+d)+\epsilon}$. In view of (3.3.8) the proof of the estimate (iii) is now complete. $\square$

3.4 Minor arcs analysis

In this section we deal with the set of minor arcs $\mathfrak{p} = ([0,1) \times \mathfrak{m}) \cup (\mathfrak{n}_\xi \times \mathfrak{M})$. Here we aim to show that for $s_{\min} \leq s \leq s_{\max}$ one has

$$\int_{\mathfrak{p}} |\mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} = o\left(P^{s-(\theta+d)}\right).$$

For a better presentation of our approach we split the analysis into two parts, dealing separately with the sets $[0,1) \times \mathfrak{m}$ and $\mathfrak{n}_\xi \times \mathfrak{M}$.

3.4.1 Minor arcs: Part 1

First we consider the case where $(\alpha_d, \alpha_\theta) \in [0,1) \times \mathfrak{m}$. Recall that the set $\mathfrak{m}$ is given by

$$\mathfrak{m} = \{\alpha_\theta \in \mathbb{R} : P^{-\theta+\delta_0} \leq |\alpha_\theta| < P^\omega\}.$$

Define the intervals $\mathfrak{m}^+ = \left[P^{-\theta+\delta_0}, P^\omega\right)$, $\mathfrak{m}^- = \left(-P^\omega, -P^{-\theta+\delta_0}\right]$ and note that $\mathfrak{m} = \mathfrak{m}^+ \cup \mathfrak{m}^-$. Recall (3.2.9). Making a change of variables by

$$\begin{pmatrix} \alpha_\theta \\ \alpha_d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_\theta \\ \beta_d \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.4.1}$$

and using the 1-periodicity of the function $\mathcal{F}(\boldsymbol{\alpha})$ with respect to α_d, yields

$$R_\pm\left(P; [0,1) \times \mathfrak{m}^-\right) = \overline{R_\pm\left(P; [0,1) \times \mathfrak{m}^+\right)}, \tag{3.4.2}$$

where $\overline{R_\pm\left(P; [0,1) \times \mathfrak{m}^+\right)}$ is the complex conjugate. Therefore, it suffices to deal with the set $[0,1) \times \mathfrak{m}^+$.

Let f be a real valued function defined on the natural numbers, and let $h \in \mathbb{N}$. Define the forward difference operator $\Delta_h f$ via the relation

$$\left(\Delta_h f\right)(x) = f(x+h) - f(x).$$

For a tuple $\mathbf{h} = (h_1, \ldots, h_t) \in \mathbb{N}^t$ we define the difference operator $\Delta_{h_1,\ldots,h_t} = \Delta_{\mathbf{h}}^{(t)}$ inductively by

$$\Delta_{\mathbf{h}}^{(t)} f(x) = \Delta_{h_t}\left(\Delta_{h_1,\ldots,h_{t-1}} f(x)\right).$$

It is apparent that the operator Δ_h is a linear one. Namely, for constants a, b, and two functions f, g, one has

$$\Delta_h (af + bg) = a\Delta_h f + b\Delta_h g.$$

For $d \geq 2$ one can inductively verify that

$$\Delta_{\mathbf{h}}^{(d)}(x^d) = d!\, h_1 \cdots h_d.$$

Next, we wish to obtain an analogous result for a monomial of fractional degree θ.

Lemma 3.4.1. *Suppose that $t \leq \lfloor \theta \rfloor$ is a natural number. Let $\boldsymbol{h} \in (\mathbb{N} \cap [1, P])^t$ and suppose that $P < x \leq 2P$. Then for each natural number $r \geq 1$ one has*

$$\left| \frac{\mathrm{d}^r}{\mathrm{d}x^r} \Delta_{\mathbf{h}}^{(t)}(x^\theta) \right| \asymp h_1 \cdots h_t P^{\theta - r - t}.$$

Proof. Observe that if $\phi : I \to \mathbb{R}$ is an r times differentiable function defined on an interval I and h is a natural number, then one has for $x_0 \in I$ that

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r} \Delta_h \phi(x)\Big|_{x=x_0} = \frac{\mathrm{d}^r}{\mathrm{d}x^r} \left(\phi(x+h) - \phi(x)\right)\Big|_{x=x_0} = \Delta_h \left(\frac{\mathrm{d}^r}{\mathrm{d}x^r} \phi(x)\Big|_{x=x_0} \right).$$

From the inductively definition of the operator $\Delta_{\mathbf{h}}^{(t)}$ and iterating we obtain from the above observation that

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r} \left(\Delta_{\mathbf{h}}^{(t)}(x^\theta) \right) = \Delta_{\mathbf{h}}^{(t)} \left(\frac{\mathrm{d}^r}{\mathrm{d}x^r}(x^\theta) \right) = C_r \Delta_{\mathbf{h}}^{(t)}(x^{\theta - r}),$$

where $C_r = \theta(\theta - 1) \cdots (\theta - r + 1)$.

From the above considerations follows that it suffices to show

$$\left| \Delta_{\mathbf{h}}^{(t)}(x^{\theta - r}) \right| \asymp h_1 \cdots h_t P^{\theta - r - t}. \tag{3.4.3}$$

To this end, we use induction on the number of shifts t and apply successively the mean value theorem of differential calculus to show that one has

$$\Delta_{\mathbf{h}}^{(t)}(x^{\theta - r}) = C_{r,t} h_1 \cdots h_d \xi_x^{\theta - r - t},$$

for some $\xi_x = \xi_{x,\mathbf{h}}$ with $x < \xi_x < x + h_1 + \cdots + h_t$, where $C_{r,t} = (\theta - r)(\theta - r - 1) \cdots (\theta - r - t + 1)$. For $t = 1$ one has

$$\Delta_{h_1}(x^{\theta - r}) = \left((x + h_1)^{\theta - r} - x^{\theta - r} \right) = (\theta - r) h_1 \xi_x^{\theta - r - 1},$$

for some $\xi_x = \xi_{x,h_1}$ with $x < \xi_x < x + h_1$. Assume that the statement of the lemma holds for $t - 1$. We prove that it does hold for t. By the definition of the forward difference operator one has

$$\Delta_{\mathbf{h}}^{(t)}(x^{\theta - r}) = \Delta_{h_t} \left(\Delta_{\mathbf{h}'}^{(t-1)}(x^{\theta - r}) \right),$$

where $\mathbf{h}' = (h_1, \ldots, h_{t-1})$.

By the inductive hypothesis one has

$$\Delta_{\mathbf{h}}^{(t-1)}(x^{\theta-r}) = (\theta - r)\cdots(\theta - r - t + 2)h_1\cdots h_{t-1}\zeta_x^{\theta-r-t+1},$$

for some $\zeta_x = \zeta_{x,\mathbf{h}'}$ with $x < \zeta_x < x + h_1 + \cdots + h_{t-1}$. We put $f(\zeta_x) = \zeta_x^{\theta-r-t+1}$ and write

$$f'(\zeta_x) = \frac{\mathrm{d}f(\zeta_x)}{\mathrm{d}\zeta_x}.$$

Clearly, $f'(\zeta_x) = (\theta - r - t + 1)\zeta_x^{\theta-r-t}$. One now has

$$\Delta_{\mathbf{h}}^{(t)}(x^{\theta-r}) = (\theta - r)\cdots(\theta - r - t + 2)h_1\cdots h_{t-1}\left(f(\zeta_x + h_t) - f(\zeta_x)\right). \tag{3.4.4}$$

To treat the expression in the parenthesis one may apply the mean value theorem of differential calculus to the function f. Hence one may write

$$f(\zeta_x + h_t) - f(\zeta_x) = (\theta - r - t + 1)h_t\xi_x^{\theta-r-t}, \tag{3.4.5}$$

for some $\xi_x = \xi_{x,\mathbf{h}}$ with $\zeta_x < \xi_x < \zeta_x + h_t$. By the induction process it is apparent that one has $x < \xi_x < x + h_1 + \cdots + h_t$. It is apparent that whenever $1 \leq \mathbf{h} \leq P$ and $P < x \leq 2P$ one has $\xi_x \asymp x \asymp P$. Putting together (3.4.4) and (3.4.5) confirms (3.4.3), and thus the proof of the lemma is complete. $\square$

In the analysis below we make use of Weyl's inequality, arising from the differencing process.

Lemma 3.4.2 (Weyl's inequality)**.** *Let $\phi(x)$ be a real valued function defined over the natural numbers. Let $d \geq 2$ be a natural number, and write $D = 2^{d-1}$. Then one has*

$$\left|\sum_{1\leq x\leq X} e(\phi(x))\right|^D \ll X^{D-1} + X^{D-d}\left|\sum_{h_1=1}^{X}\cdots\sum_{h_{d-1}=1}^{X}\sum_{1\leq x<x+Y_{d-1}\leq X} e\left(\Delta_{\mathbf{h}}^{(d-1)}(\phi(x))\right)\right|,$$

where $Y_j = h_1 + \cdots h_j$, for each j. The implied constant depends only on d, and an empty sum denotes zero.

Proof. See [2, Lemma 3.8]. $\square$

From now one we fix an index i. By Lemma 3.4.2, and using the linearity of the forward difference operator one has

$$\begin{aligned}|F_i(\alpha_d, \alpha_\theta)|^{2^d} &\ll P^{2^d-1} + P^{2^d-(d+1)}\sum_{\mathbf{h}}\left|\sum_x e\left(a_i\alpha_d d!\, h_1\cdots h_d + \lambda_i\alpha_\theta\Delta_{\mathbf{h}}^{(d)}(x^\theta)\right)\right| \\ &\ll P^{2^d-1} + P^{2^d-(d+1)}\sum_{\mathbf{h}}\left|\sum_x e\left(\lambda_i\alpha_\theta\Delta_{\mathbf{h}}^{(d)}(x^\theta)\right)\right|,\end{aligned}$$

where in the second step we used the triangle inequality. In the above summation notation, we sum over tuples $\mathbf{h}$ satisfying $1 \leq \mathbf{h} \leq P$ and x belongs to a subinterval of $[1, P]$ determined by the shifts $h_1, \ldots, h_d$. For convenience we denote this interval by $I(\mathbf{h})$.

We define the exponential

$$S_i(\alpha_\theta, \mathbf{h}) = \sum_{x \in I(\mathbf{h})} e\left(\lambda_i \alpha_\theta \Delta_{\mathbf{h}}^{(d)}(x^\theta)\right). \tag{3.4.6}$$

Hence, the above estimate now takes the shape

$$|F_i(\alpha_d, \alpha_\theta)|^{2^d} \ll P^{2^d-1} + P^{2^d-(d+1)} \sum_{\mathbf{h}} |S_i(\alpha_\theta, \mathbf{h})|. \tag{3.4.7}$$

One can split the summation over $\mathbf{h}$ based on the size of the product $H = h_1 \cdots h_d$. Consider the function $\psi(P) = (\log P)^{-1}$ which decreases monotonically to zero as $P \to \infty$ and furthermore for large P satisfies $\psi(P) > P^{-\epsilon}$ for any fixed $\epsilon > 0$. We form a partition of the shape

$$\{(h_1, \dots, h_d) : h_i \in [1, P] \cap \mathbb{Z}\} = A_1 \cup A_2 \cup A_3,$$

where we define the sets A_1, A_2 and A_3 by

$$A_1 = \left\{(h_1, \dots, h_d) : h_i \in [1, P] \cap \mathbb{Z},\ P^d \psi(P) < H \leq P^d\right\},$$

$$A_2 = \left\{(h_1, \dots, h_d) : h_i \in [1, P] \cap \mathbb{Z},\ P^{d-5^{-\theta}} < H \leq P^d \psi(P)\right\},$$

$$A_3 = \left\{(h_1, \dots, h_d) : h_i \in [1, P] \cap \mathbb{Z},\ H \leq P^{d-5^{-\theta}}\right\}.$$

Moreover, for $\kappa = 1, 2, 3$ we define

$$T_\kappa(\alpha_\theta) = \sum_{\mathbf{h} \in A_\kappa} |S_i(\alpha_\theta, \mathbf{h})|. \tag{3.4.8}$$

Hence, we may now write

$$\sum_{\mathbf{h}} |S_i(\alpha_\theta, \mathbf{h})| \ll T_1(\alpha_\theta) + T_2(\alpha_\theta) + T_3(\alpha_\theta).$$

Invoking (3.4.7) we deduce that

$$|F_i(\alpha_d, \alpha_\theta)|^{2^d} \ll P^{2^d-1} + P^{2^d-(d+1)} \left(T_1(\alpha_\theta) + T_2(\alpha_\theta) + T_3(\alpha_\theta)\right). \tag{3.4.9}$$

Our aim now is to obtain a non-trivial upper bound for the exponential sum $S_i(\alpha_\theta)$ with $\alpha_\theta \in \mathfrak{m}^+$. To do so, we make use of van der Corput's k-th derivative test (see Lemma 2.4.1) for bounding exponential sums. We now make some observations that set the ground for an application of Lemma 2.4.1. It is convenient to work with an exponential sum over a dyadic interval. Recall from (3.4.6) that

$$S_i(\alpha_\theta, \mathbf{h}) = \sum_{x \in I(\mathbf{h})} e\left(\lambda_i \alpha_\theta \Delta_{\mathbf{h}}^{(d)}(x^\theta)\right).$$

One may split the interval $I(\mathbf{h})$ into $O(\log P)$ dyadic intervals. By making abuse of notation one

then has
$$|S_i(\alpha_\theta, \mathbf{h})| \ll \log P \sum_{P<x\leq 2P} e\left(\lambda_i \alpha_\theta \Delta_{\mathbf{h}}^{(d)}(x^\theta)\right).$$

Define the exponential sum

$$\widetilde{S}_i(\alpha_\theta, \mathbf{h}) = \sum_{P<x\leq 2P} e\left(\lambda_i \alpha_\theta \Delta_{\mathbf{h}}^{(d)}(x^\theta)\right).$$

Hence for all α_θ and for any fixed $\epsilon > 0$ one has

$$|S_i(\alpha_\theta, \mathbf{h})| \ll P^\epsilon \left|\widetilde{S}_i(\alpha_\theta, \mathbf{h})\right|. \tag{3.4.10}$$

It is apparent that an upper bound for the exponential sum $\widetilde{S}_i(\alpha_\theta)$ leads to an upper bound for the exponential sum $S_i(\alpha_\theta)$ with an ϵ- loss. This is enough for our purpose. Observe that invoking Lemma 3.4.1 with $t = d$ one has for each natural number $r \geq 1$ that

$$\left|\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\lambda_i \alpha_\theta \Delta_{\mathbf{h}}^{(d)}(x^\theta)\right)\right| \asymp FP^{-r},$$

where $F = |\lambda_i C_d C_{d,r}||\alpha_\theta| H P^{\theta-d}$. Recall here that $\mathfrak{m}^+ = \left[P^{-\theta+\delta_0}, P^\omega\right)$.

Lemma 3.4.3. *Suppose that $P^{d-5^{-\theta}} < H \leq P^d$. For each index i and for any $\alpha_\theta \in \mathfrak{m}^+$ one has that*
$$|S_i(\alpha_\theta, \boldsymbol{h})| \ll P^{1-4^{-\theta}}.$$

Proof. Note that it is enough to show that for all $\alpha_\theta \in \mathfrak{m}^+$ one has

$$\left|\widetilde{S}_i(\alpha_\theta, \mathbf{h})\right| \ll P^{1-\sigma},$$

for some $\sigma > 4^{-\theta}$. Then returning in (3.4.10) and taking $\epsilon = \sigma - 4^{-\theta} > 0$ as we are at liberty to do, yields the desired conclusion. We consider two separate cases depending on the size of H.

Suppose first that $P^d \psi(P) < H \leq P^d$. Then one has

$$P^{\delta_0} \psi(P) \ll F \ll P^{\theta+\omega}.$$

We may now apply Lemma 2.4.1 with $q = n$, where temporarily we write $n = \lfloor \theta \rfloor$. This reveals that for any $\alpha_\theta \in \mathfrak{m}^+$ one has

$$\left|\widetilde{S}_i(\alpha_\theta, \mathbf{h})\right| \ll P^{1-\sigma} + P^{1-\delta_0} \psi(P)^{-1},$$

where
$$\sigma = \frac{n + 2 - \theta - \omega}{2^{n+2} - 2}. \tag{3.4.11}$$

Recalling (3.2.7) one may verify that for $\theta > d + 1 \geq 3$ one has

$$\sigma > \frac{1}{3^\theta + 6} > \frac{1}{4^\theta}.$$

Moreover, recalling that $\psi(P) = (\log P)^{-1}$ one has $\psi(P)^{-1} \ll P^{10^{-\theta}}$, which yields

$$P^{1-\delta_0}\psi(P)^{-1} \ll P^{1-\delta_0+10^{-\theta}}.$$

Hence, the previous estimate for the exponential sum $\widetilde{S}_i(\alpha_\theta, \mathbf{h})$ delivers

$$\left|\widetilde{S}_i(\alpha_\theta, \mathbf{h})\right| \ll P^{1-\sigma'},$$

where $\sigma' = \min\{\sigma,\, \delta_0 - 10^{-\theta}\} > 4^{-\theta}$ and we are done.

Suppose now that $P^{d-5^{-\theta}} < H \le P^d\psi(P)$. In this case one has

$$P^{\delta_0 - 5^{-\theta}} \ll F \ll P^{\theta+\omega}\psi(P).$$

Applying again Lemma 2.4.1 with $q = n$, yields that for any $\alpha_\theta \in \mathfrak{m}^+$ one has

$$\left|\widetilde{S}_i(\alpha_\theta, \mathbf{h})\right| \ll P^{1-\sigma}\left(\psi(P)\right)^{1/(2^{n+2}-2)} + P^{1-\delta_0+5^{-\theta}},$$

with σ as in (3.4.11). For large P one may assume that $\psi(P) < 1$. Recalling again from (3.2.7) that $\delta_0 = 2^{1-2\theta}$ the above estimate delivers

$$\left|\widetilde{S}_i(\alpha_\theta, \mathbf{h})\right| \ll P^{1-\sigma'},$$

where now we write $\sigma' = \min\{\sigma,\, \delta_0 - 5^{-\theta}\} > 4^{-\theta}$. Thus the proof is now complete. □

We may now estimate the sums $T_\kappa(\alpha_\theta)$ $(1 \le \kappa \le 3)$ defined in (3.4.8).

Lemma 3.4.4. *For each index i and for any $\alpha_\theta \in \mathfrak{m}^+$ one has that*

(i) $|T_1(\alpha_\theta)| \ll P^{d+1-5^{-\theta}}$;

(ii) $|T_2(\alpha_\theta)| \ll P^{d+1-5^{-\theta}}\psi(P)$;

(iii) $|T_3(\alpha_\theta)| \ll P^{d+1-6^{-\theta}}$.

Proof. For each $\kappa = 1, 2, 3$ we write $\#A_\kappa$ to denote the cardinality of the set A_κ. We set

$$X_1 = P^d, \qquad X_2 = P^d\psi(P), \qquad X_3 = P^{d-5^{-\theta}}.$$

Observe that for each $\kappa = 1, 2, 3$ and for any fixed $\epsilon > 0$ one has

$$\#A_\kappa \ll \sum_{H \le X_\kappa} \tau_d(H) \ll X_\kappa P^\epsilon,$$

where recall that $H = h_1 \cdots h_d$ and τ_d is the d-fold divisor function.

One may get an upper bound for each $T_\kappa(\alpha_\theta)$ by using the above observation together with the bound supplied by Lemma 3.4.3. Let us demonstrate this by proving estimate (i). Recall here that

$$T_1(\alpha_\theta) = \sum_{\mathbf{h}\in A_1} |S_i(\alpha_\theta, \mathbf{h})|,$$

where

$$A_1 = \{(h_1,\dots,h_d) : h_i \in [1,P]\cap\mathbb{Z},\;\; P^d\psi(P) < H \le P^d\}.$$

Invoking Lemma 3.4.3 one has for any $\alpha_\theta \in \mathfrak{m}^+$ and any fixed $\epsilon > 0$ that

$$|T_1(\alpha_\theta)| \ll \left(\sup_{\substack{\alpha_\theta\in\mathfrak{m}^+\\ \mathbf{h}\in A_1}} |S_i(\alpha_\theta,\mathbf{h})|\right)\sum_{\mathbf{h}\in A_1} 1 \ll P^{1-4^{-\theta}}\,(\#A_1) \ll P^{d+1-4^{-\theta}+\epsilon}.$$

Pick now a sufficiently small $0<\epsilon<4^{-\theta}-5^{-\theta}$ to deduce that for any $\alpha_\theta \in \mathfrak{m}^+$ one has

$$|T_1(\alpha_\theta)| \ll P^{d+1-5^{-\theta}}.$$

Similarly we may argue to estimate the sums $T_2(\alpha_\theta)$ and $T_3(\alpha_\theta)$. For the sake of clarity, let us mention that in estimating $T_3(\alpha_\theta)$ one may use the trivial bound

$$|S_i(\alpha_\theta,\mathbf{h})| \ll P,$$

which is always valid. With this observation the proof of the lemma is now complete. □

By Lemma 3.4.4 it is apparent that for each index i and for any $\alpha_\theta \in \mathfrak{m}^+$ one has

$$|T_\kappa(\alpha_\theta)| \ll P^{d+1-6^{-\theta}} \qquad (\kappa = 1,2,3).$$

One may now use the above estimate in order to bound from above the right hand side of (3.4.9). Hence we deduce that

$$|F_i(\alpha_d,\alpha_\theta)| \ll P^{1-1/2^d} + P^{1-1/(2^d\cdot 6^\theta)} \ll P^{1-6^{-\theta-d}}.$$

Upon recalling (3.2.5) we have proved the following.

Lemma 3.4.5. *For each index i and for any $(\alpha_d,\alpha_\theta)\in[0,1)\times\mathfrak{m}^+$ one has that*

$$|f_i(\alpha_d,\alpha_\theta)| \ll P^{1-6^{-\theta-d}}.$$

Equipped with all the necessary auxiliary estimates, we may now finish up the first part of the minor arcs analysis. We now set

$$\eta_1 = 6^{-\theta-d} \qquad \text{and} \qquad \kappa = P^\omega.$$

Note that for large enough P one has $\min_{i,j}\{\kappa|\lambda_i|,\;\kappa|\mu_j|\} \ge 1$. Recall from (3.2.11) that one has $s' = s-\delta$ and recall as well from (3.2.15) that $s' = A_\theta + (1-\omega_1)A_d$. One may now use Lemma 3.4.5 and Lemma 3.3.6 in order to estimate the right hand side of the inequality (3.2.14). Hence, we may infer that for any fixed $\epsilon>0$ one has

$$\int_{\mathfrak{m}^+}\int_0^1 |\mathcal{F}(\boldsymbol{\alpha})K_\pm(\alpha_\theta)|\,\mathrm{d}\boldsymbol{\alpha} \ll P^{(1-\eta_1)\delta}P^{s'-(\theta+d)+\omega+\epsilon} \ll P^{s-(\theta+d)-\eta_1\delta+\omega+\epsilon}.$$

Recall from(3.2.7) that $\omega \leq 5^{-100(\theta+d)}$. One may choose

$$\delta = 6^{-\theta} \in (0, 1/3) \qquad \text{and} \qquad \epsilon = 5^{-100(\theta+d)},$$

as we are at liberty to do. With these choices for δ and ϵ it is clear that $-\eta_1\delta + \omega + \epsilon < 0$. Thus the above estimate delivers

$$\int_{\mathfrak{m}^+} \int_0^1 |\mathcal{F}(\boldsymbol{\alpha}) K_{\pm}(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} = o\left(P^{s-(\theta+d)}\right).$$

In the light of (3.4.2) we have established the following.

Lemma 3.4.6. *For $s_{\min} \leq s \leq s_{\max}$ one has*

$$\int_{\mathfrak{m}} \int_0^1 |\mathcal{F}(\boldsymbol{\alpha}) K_{\pm}(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} = o\left(P^{s-(\theta+d)}\right).$$

3.4.2 Minor arcs: Part 2

In this subsection we consider the case where $(\alpha_d, \alpha_\theta) \in \mathfrak{n}_\xi \times \mathfrak{M}$. Let us recall here that $\mathfrak{n}_\xi \subset [0, 1)$ is a set of minor arcs in the classical sense and $\mathfrak{M} = \left(-P^{-\theta+\delta_0}, P^{-\theta+\delta_0}\right)$. We put $\mathfrak{M}^+ = \left(0, P^{-\theta+\delta_0}\right)$ and $\mathfrak{M}^- = \left(-P^{-\theta+\delta_0}, 0\right)$. Note that $\mathfrak{M} = \mathfrak{M}^+ \cup \mathfrak{M}^-$. Making a change of variables as in (3.4.1) yields

$$R_{\pm}\left(P; \mathfrak{n}_\xi \times \mathfrak{M}^-\right) = \overline{R_{\pm}\left(P; \mathfrak{n}_\xi \times \mathfrak{M}^+\right)}. \tag{3.4.12}$$

So in the following it suffices to deal with the set $\mathfrak{n}_\xi \times \mathfrak{M}^+$. The point of departure in our approach is the following version of the Weyl - van der Corput inequality.

Lemma 3.4.7 (Weyl–van der Corput inequality)**.** *Suppose that I is a finite subset of $\mathbb{N}$, and suppose that $(w(n))_{n\in\mathbb{N}} \subset \mathbb{C}$ is a complex-valued sequence, such that $w(n) = 0$ for $n \notin I$. Let H be a positive integer. Then one has,*

$$\left|\sum_{n\in\mathbb{N}} w(n)\right|^2 \leq \frac{\mathrm{card}(I) + H}{H} \sum_{|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in\mathbb{N}} w(n)\overline{w(n-h)}.$$

Proof. See [43, Lemma 2.5]. □

To begin with, let us fix an index i. Apply Lemma 3.4.7 to the exponential sum $F_i(\alpha_d, \alpha_\theta)$, with $I = [1, P] \cap \mathbb{N}$. For an integer $H \in [1, P]$ with $H = o(P)$ to be chosen at a later stage one has

$$|F_i(\alpha_d, \alpha_\theta)|^2 \ll \frac{P+H}{H} \sum_{|h|<H} \sum_{1\leq x\leq P} e\left(a_i\alpha_d\Delta_h\left(x^d\right) + \lambda_i\alpha_\theta\Delta_h\left(x^\theta\right)\right). \tag{3.4.13}$$

By the mean value theorem of differential calculus one has that

$$|(x+h)^\theta - x^\theta| \asymp |h|P^{\theta-1} \ll HP^{\theta-1}.$$

For $\alpha_\theta \in \mathfrak{M}^+$ the above estimate leads to

$$|\alpha_\theta||(x+h)^\theta - x^\theta| \ll P^{-1+\delta_0}H.$$

Using the elementary inequality $|e(x)| \leq 2\pi|x|$ which is valid for all $x \in \mathbb{R}$, we may infer that for any $\alpha_\theta \in \mathfrak{M}^+$ one has

$$\left|e\left(\lambda_i\alpha_\theta\Delta_h\left(x^\theta\right)\right)\right| \ll P^{-1+\delta_0}H.$$

One may now use the fact that $|e(x)| \leq 1$ for all $x \in \mathbb{R}$, together with the above estimate to derive that

$$\sum_{|h|<H}\sum_{1\leq x\leq P} e\left(a_i\alpha_d\Delta_h\left(x^d\right)+\lambda_i\alpha_\theta\Delta_h\left(x^\theta\right)\right) = \sum_{|h|<H}\sum_{1\leq x\leq P} e\left(a_i\alpha_d\Delta_h\left(x^d\right)\right) + O\left(P^{\delta_0}H^2\right).$$

Substituting the above conclusion into (3.4.13) and using the fact that $H = o(P)$ yields

$$|F_i(\alpha_d,\alpha_\theta)|^2 \ll P^{1+\delta_0}H + \frac{P+H}{H}\sum_{|h|<H}|W_i(\alpha_d,h)|, \tag{3.4.14}$$

where we write

$$W_i(\alpha_d,h) = \sum_{1\leq x\leq P} e\left(a_i\alpha_d\Delta_h\left(x^d\right)\right).$$

We now examine separately the cases $d \geq 3$ and $d = 2$.

First we consider the case $d \geq 3$. An application of Hölder's inequality reveals

$$\left(\sum_{|h|<H}|W_i(\alpha_d,h)|\right)^{2^{d-2}} \ll H^{2^{d-2}-1}\sum_{|h|<H}|W_i(\alpha_d,h)|^{2^{d-2}}. \tag{3.4.15}$$

Applying Weyl's differencing process, we infer by Lemma 3.4.2 that

$$|W_i(\alpha_d,h)|^{2^{d-2}} \ll P^{2^{d-2}-1} + P^{2^{d-2}-(d-1)}\sum_{\mathbf{h}}\left|\sum_{x\in I(\mathbf{h})} e\left(d!\,hh_1\cdots h_{d-2}a_i\alpha_d x\right)\right|,$$

where in the above summation notation, we sum over tuples $\mathbf{h} = (h_1,\ldots,h_{d-2})$ satisfying $1 \leq \mathbf{h} \leq P$ and $I(\mathbf{h})$ is a subinterval of $[1,P]$, determined by the shifts $h_1,\ldots,h_{d-2}$.

Invoking a classical estimate for the sum of the geometric series we see that

$$\left|\sum_{x\in I(\mathbf{h})} e\left(d!\,hh_1\cdots h_{d-2}\alpha_d a_i x\right)\right| \ll \min\left\{P,\ \|d!\,hh_1\cdots h_{d-2}a_i\alpha_d\|^{-1}\right\}.$$

Hence by the preceding estimate concerning $W_i(\alpha_d,h)$ we deduce that

$$\sum_{|h|<H}|W_i(\alpha_d,h)|^{2^{d-2}} \ll HP^{2^{d-2}-1} + P^{2^{d-2}-(d-1)} \times$$

$$\times\sum_{h=1}^{H}\sum_{h_1=1}^{P}\cdots\sum_{h_{d-2}=1}^{P}\min\left\{P, \|d!\,hh_1\cdots h_{d-2}a_i\alpha_d\|^{-1}\right\}.$$

We write $d!\,|a_i|hh_1\cdots h_{d-2} = m$. Note that for $1 \leq h \leq H$ and for $\mathbf{h} = (h_1,\ldots,h_{d-2})$ with $1 \leq \mathbf{h} \leq P$ one has that $m \in \mathbb{Z}\cap[1,\ d!\,|a_i|HP^{d-2}]$. Clearly, the number of solutions of the previous equation with respect to m is $\leq \tau_{d-1}(m) \ll_{d,a_i} m^\epsilon$. Thus, for any fixed $0<\epsilon<1$ we obtain

$$\sum_{|h|<H} |W_i(\alpha_d, h)|^{2^{d-2}} \ll HP^{2^{d-2}-1} + P^{2^{d-2}-(d-1)+\epsilon} \sum_{m=1}^{d!\,|a_i|HP^{d-2}} \min\left\{P, \|m\alpha_d\|^{-1}\right\}. \tag{3.4.16}$$

We may bound the sum on the right hand side of the above estimate using the following.

Lemma 3.4.8. *Suppose that α, β are real numbers and suppose further that $|\alpha - a/q| \leq 1/q^2$, where $(a, q) = 1$. Then*

$$\sum_{z=1}^{R} \min\{N, \|\alpha z + \beta\|\} \ll (N + q\log q)\left(\frac{R}{q} + 1\right).$$

Proof. See [2, Lemma 3.2]. For the sake of clarity we remark here that in the statement Baker is imposing a strict inequality, namely $|\alpha - a/q| < 1/q^2$. However it is apparent from the proof that this is unnecessary. □

By Dirichlet's theorem on Diophantine approximation, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ which satisfy $(a, q) = 1,\ 1 \leq q \leq HP^{d-1-\xi}$ and

$$\left| a_i\alpha_d - \frac{a}{q} \right| \leq \frac{1}{qHP^{d-1-\xi}}.$$

We pause for a moment to reflect on the fact that $\alpha_d \in \mathfrak{n}_\xi$. Recall that we assume $H = o(P)$. So for large enough P one has $HP^{d-1-\xi} < P^{d-\xi}$. So if it was $1 \leq q \leq P^{\xi}$, then α_d would belong to the set of major arcs $\mathfrak{N}_\xi$. Thus, we may suppose that $q > P^{\xi}$. Hence

$$P^{\xi} < q \leq HP^{d-1-\xi}. \tag{3.4.17}$$

One may now apply Lemma 3.4.8. For any fixed $0 < \epsilon < 1$ one has

$$\begin{aligned}\sum_{m=1}^{d!\,|a_i|HP^{d-2}} \min\left\{P, \|m\alpha_d\|^{-1}\right\} &\ll (P + q\log q)\left(\frac{d!\,|a_i|HP^{d-2}}{q} + 1\right) \\ &\ll HP^{d-1+\epsilon}\left(\frac{1}{q} + \frac{1}{P} + \frac{q}{HP^{d-1}}\right),\end{aligned}$$

where in the second step estimate, we used the facts that $\log q \ll P^{\epsilon}$, and that for $d \geq 3$ one has $HP^{d-2}\log q \gg P$. By (3.4.17) one has

$$\frac{1}{q} + \frac{1}{P} + \frac{q}{HP^{d-1}} \ll P^{-\xi}.$$

Thus, the previous estimates delivers

$$\sum_{m=1}^{d!\,|a_i|HP^{d-2}} \min\left\{P, \|m\alpha_d\|^{-1}\right\} \ll HP^{d-1-\xi+\epsilon}.$$

Using the above bound, one may now estimate the right hand side of (3.4.16) to obtain

$$\sum_{|h|<H} |W_i(\alpha_d, h)|^{2^{d-2}} \ll HP^{2^{d-2}-1} + HP^{2^{d-2}-\xi+\epsilon}.$$

Invoking (3.4.15) the previous estimate implies

$$\sum_{|h|<H} |W_i(\alpha_d, h)| \ll HP^{1-\xi/2^{d-2}+\epsilon}.$$

Incorporating the above into (3.4.14) and using the fact that $H = o(P)$, yields that for any $\alpha_d \in \mathfrak{n}_\xi$ one has

$$|F_i(\alpha_d, \alpha_\theta)|^2 \ll P^{1+\delta_0}H + P^{2-\xi/2^{d-2}+\epsilon}. \tag{3.4.18}$$

We now deal with the case where $d = 2$. In this case one does not have to apply Weyl's differencing process. Note that for $d = 2$ the difference $\Delta_h(x^2) = 2xh + h^2$ is already a linear polynomial with respect to x. So one has

$$|W_i(\alpha_d, h)| \leq \left| \sum_{1 \leq x \leq P} e\left(2ha_i\alpha_d x\right) \right| \ll \min\left\{P, \|2ha_i\alpha_d\|^{-1}\right\}.$$

Thus,

$$\sum_{|h|<H} |W_i(\alpha_d, h)| \ll \sum_{m=1}^{2|a_i|H} \min\left\{P, \|m\alpha_d\|^{-1}\right\}.$$

One may now apply Dirichlet's theorem on Diophantine approximation and argue as in the case $d \geq 3$. Here the inequality (3.4.17) is replaced by $P^\xi < q \leq HP^{1-\xi}$. Applying Lemma 3.4.8 one has

$$\begin{aligned}\sum_{m=1}^{2|a_i|H} \min\left\{P, \|2ha_i\alpha_d\|^{-1}\right\} &\ll (P + q\log q)\left(\frac{2|a_i|H}{q} + 1\right) \\ &\ll HP^{1-\xi+\epsilon} + P^{1+\epsilon},\end{aligned}$$

where in the second step estimate we used the facts that $P \gg H \log q$ and $H \ll HP$. Therefore, by (3.4.14) we infer that

$$|F_i(\alpha_d, \alpha_\theta)|^2 \ll P^{1+\delta_0}H + P^{2-\xi+\epsilon} + P^{2+\epsilon}H^{-1}. \tag{3.4.19}$$

We may now obtain a non-trivial upper bound for the exponential sum $F_i(\alpha_d, \alpha_\theta)$. Recall that $H \in [1, P]$ is an integer at our disposal which satisfies $H = o(P)$. Let us now choose a value for H so that $H \asymp P^\varpi$ where $\varpi = (1 - \delta_0)/2 \in (0, 1)$. First we deal with the case $d \geq 3$. Recall from (3.2.7) that $\delta_0 = 2^{1-2\theta}$ and recall from (3.2.8) that $0 < \xi \leq \delta_0/8$. By (3.4.18) we deduce that for any fixed $0 < \epsilon < 1$ and any $\alpha_d \in \mathfrak{n}_\xi$ one has that

$$|F_i(\alpha_d, \alpha_\theta)| \ll P^{1-\xi/2^{d-3}+\epsilon}.$$

Now we come to the case $d = 2$. With the above choice for the integer parameter H we may infer by (3.4.19) that for any fixed $0 < \epsilon < 1$ and any $\alpha_d \in \mathfrak{n}_\xi$ one has

$$|F_i(\alpha_d, \alpha_\theta)| \ll P^{1-\xi/2+\epsilon}.$$

By the preceding conclusions and recalling (3.2.5) we have proved the following.

Lemma 3.4.9. *For each index i and for any $(\alpha_d, \alpha_\theta) \in \mathfrak{n}_\xi \times \mathfrak{M}$, one has for any fixed $0 < \epsilon < 1$ that*

$$|f_i(\alpha_d, \alpha_\theta)| \ll \begin{cases} P^{1-\xi/2+\epsilon}, & \textit{when} \quad d = 2, \\ \\ P^{1-\xi/2^{d-3}+\epsilon}, & \textit{when} \quad d \geq 3. \end{cases}$$

We may now finish our analysis as in Part 1 of the minor arcs treatment. Below we demonstrate how to deal with the case $d \geq 3$. One may argue similarly when $d = 2$. Put

$$\eta_2 = \xi/2^{d-3} \qquad \text{and} \qquad \kappa = \max_{i,j} \left\{ |\lambda_i|^{-1}, \ |\mu_j|^{-1} \right\}.$$

Note that now κ is a fixed real number such that $\min_{i,j}\{\kappa|\lambda_i|, \ \kappa|\mu_j|\} \geq 1$. As in Part 1 of the minor arcs analysis, one may now use Lemma 3.4.9 and Lemma 3.3.6 in order to estimate the right hand side of the inequality (3.2.14). Hence, we may infer that for any fixed $0 < \epsilon < 1$ one has

$$\int_{\mathfrak{M}^+} \int_{\mathfrak{n}_\xi} |\mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} \ll P^{(1-\eta_2+\epsilon)\delta} \cdot P^{s'-(\theta+d)+\epsilon} \ll P^{s-(\theta+d)-\eta_2\delta+(1+\delta)\epsilon}.$$

One may now choose

$$\delta = \frac{1}{6} \in (0, 1/3) \qquad \text{and} \qquad \epsilon = \frac{\xi}{(1+\delta)2^d} \in (0,1),$$

as we are at liberty to do. With these choices one has $-\eta_2\delta + (1+\delta)\epsilon < 0$. Hence, the previous estimate delivers

$$\int_{\mathfrak{M}^+} \int_{\mathfrak{n}_\xi} |\mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} = o\left(P^{s-(\theta+d)}\right).$$

In the light of (3.4.12) we have established the following.

Lemma 3.4.10. *For $s_{\min} \leq s \leq s_{\max}$ one has*

$$\int_{\mathfrak{M}} \int_{\mathfrak{n}_\xi} |\mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} = o\left(P^{s-(\theta+d)}\right).$$

Before we close this section, we find it appropriate to record the following lemma which concerns the complete set of minor arcs

$$\mathfrak{p} = ([0,1) \times \mathfrak{m}) \cup (\mathfrak{n}_\xi \times \mathfrak{M}).$$

Combining Lemma 3.4.6 and Lemma 3.4.10 we have established the following.

Lemma 3.4.11. *For $s_{\min} \leq s \leq s_{\max}$ one has*

$$\int_{\mathfrak{p}} |\mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha} = o\left(P^{s-(\theta+d)}\right).$$

3.5 Trivial arcs

In this section we deal with the disposal of the set of trivial arcs $\mathfrak{c} = [0,1) \times \mathfrak{t}$, where recall that $\mathfrak{t} = \{\alpha_\theta \in \mathbb{R} : |\alpha_\theta| \geq P^\omega\}$. We put $\mathfrak{t}^+ = [P^\omega, \infty)$ and $\mathfrak{t}^- = (-\infty, -P^\omega]$. Note that $\mathfrak{t} = \mathfrak{t}^+ \cup \mathfrak{t}^-$. We set $\mathfrak{c}^+ = [0,1) \times [P^\omega, \infty)$ and $\mathfrak{c}^- = [0,1) \times (-\infty, P^\omega]$. By a change of variables as in (3.4.1) one has

$$R_\pm\left(P; \mathfrak{c}^-\right) = \overline{R_\pm\left(P; \mathfrak{c}^+\right)}. \tag{3.5.1}$$

So, it is enough to deal with the set $\mathfrak{c}^+$.

Fix an index i. One has

$$\mathfrak{c}^+ \subset \bigcup_{\rho = \lfloor \omega \log_2 P \rfloor}^{\infty} \left([0,1) \times \left(2^\rho, 2^{\rho+1}\right]\right).$$

We take $\kappa = 2^{\rho+1}$. Here we consider large enough values of P so that for $\rho \geq \lfloor \omega \log_2 P \rfloor$ one has $\min_{i,j}\{\kappa|\lambda_i|,\ \kappa|\mu_j|\} \geq 1$. By Lemma 3.3.6 and taking into account (3.2.2), one has for any fixed $\epsilon > 0$ that

$$\begin{aligned}
\Xi_{f_i}(\mathfrak{c}^+) &\ll \sum_{\rho = \lfloor \omega \log_2 P \rfloor}^{\infty} \int_{2^\rho}^{2^{\rho+1}} \int_0^1 f_i^{A_\theta} |K_\pm(\alpha_\theta)| \mathrm{d}\boldsymbol{\alpha} \\
&\ll P^{A_\theta - (\theta+d) + \epsilon} \sum_{\rho = \lfloor \omega \log_2 P \rfloor}^{\infty} \frac{1}{2^\rho}.
\end{aligned}$$

Clearly,

$$\sum_{\rho = \lfloor \omega \log_2 P \rfloor}^{\infty} \frac{1}{2^\rho} \ll P^{-\omega}.$$

Hence, by choosing $\epsilon = \frac{\omega}{2} > 0$ the previous estimate now delivers

$$\Xi_{f_i}(\mathfrak{c}^+) \ll P^{A_\theta - (\theta+d) - \frac{\omega}{2}}.$$

One may deal with the auxiliary mean values $\Xi_{f_i,g_j}(\mathfrak{c}^+), \Xi_{f_i,h_k}(\mathfrak{c}^+), \Xi_{g_j,h_k}(\mathfrak{c}^+)$ similarly. We may now put these estimates together. One is at liberty to take $\delta = 0$ in the inequality (3.2.14). So, in this case by (3.2.11) one has $s' = s$, and by (3.2.15) one has $s = A_\theta + (1 - \omega_1)A_d$. Thus we obtain

$$\int_{\mathfrak{t}^+} \int_0^1 |\mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} \ll P^{A_\theta + (1-\omega_1)A_d - (\theta+d) - \frac{\omega}{2}} = o\left(P^{s-(\theta+d)}\right).$$

In the light of (3.5.1) we have established the following.

Lemma 3.5.1. *For $s_{\min} \leq s \leq s_{\max}$ one has*

$$\int_{\mathfrak{c}} |\mathcal{F}(\boldsymbol{\alpha}) K_\pm(\alpha_\theta)| \, \mathrm{d}\boldsymbol{\alpha} = o\left(P^{s-(\theta+d)}\right).$$

3.6 Major arcs analysis

In this section we deal with the set of major arcs $\mathfrak{P} = \mathfrak{n}_\xi \times \mathfrak{M}$. We split the analysis into two subsections.

3.6.1 Singular integral analysis

Here we deal with the singular integral. For each index i, j and k, and any $\boldsymbol{\beta} = (\beta_d, \beta_\theta) \in \mathbb{R}^2$ we define the continuous generating functions

$$\upsilon_{f,i}(\boldsymbol{\beta}) = \int_{\frac{1}{2}x_i^\star P}^{2x_i^\star P} e(a_i\beta_d\gamma^d + \lambda_i\beta_\theta\gamma^\theta)\mathrm{d}\gamma, \qquad \upsilon_{g,j}(\boldsymbol{\beta}) = \int_{\frac{1}{2}y_j^\star P}^{2y_j^\star P} e(\mu_j\beta_\theta\gamma^\theta)\mathrm{d}\gamma,$$
$$\upsilon_{h,k}(\boldsymbol{\beta}) = \int_{\frac{1}{2}z_k^\star P}^{2z_k^\star P} e(b_k\beta_d\gamma^d)\mathrm{d}\gamma. \tag{3.6.1}$$

Moreover we write

$$V(\boldsymbol{\beta}) = \prod_{i=1}^{\ell} \upsilon_{f,i}(\boldsymbol{\beta}) \prod_{j=1}^{m} \upsilon_{g,j}(\boldsymbol{\beta}) \prod_{k=1}^{n} \upsilon_{h,k}(\boldsymbol{\beta}).$$

Define the truncated singular integrals

$$\mathfrak{J}^{\pm}(P^\xi, P^{\delta_0}) = \int_{-P^{-\theta+\delta_0}}^{P^{-\theta+\delta_0}} \int_{-P^{-d+\xi}}^{P^{-d+\xi}} V(\boldsymbol{\beta})K_{\pm}(\beta_\theta)\mathrm{d}\boldsymbol{\beta},$$
$$\mathfrak{J}(P^\xi, P^{\delta_0}) = \int_{-P^{-\theta+\delta_0}}^{P^{-\theta+\delta_0}} \int_{-P^{-d+\xi}}^{P^{-d+\xi}} V(\boldsymbol{\beta})\mathrm{d}\boldsymbol{\beta}, \tag{3.6.2}$$

and the complete singular integral

$$\mathfrak{J}(\infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\boldsymbol{\beta})\mathrm{d}\boldsymbol{\beta}. \tag{3.6.3}$$

Lemma 3.6.1. *For each index i, j, k and for any $\boldsymbol{\beta} = (\beta_d, \beta_\theta) \in \mathbb{R}^2$ one has*

$$\upsilon_{f,i}(\boldsymbol{\beta}) \ll P(1 + P^d|\beta_d| + P^\theta|\beta_\theta|)^{-1/\theta}, \quad \upsilon_{g,j}(\boldsymbol{\beta}) \ll P(1 + P^\theta|\beta_\theta|)^{-1/\theta},$$
$$\upsilon_{h,k}(\boldsymbol{\beta}) \ll P(1 + P^d|\beta_d|)^{-1/d}.$$

In the case where $\theta \in \mathbb{N}$ one can find a proof of this lemma in [78, Theorem 7.3]. In our case one has $\theta \notin \mathbb{N}$. For this reason we give an alternative proof using van der Corput's estimate for oscillatory integrals, dating back to 1935 in van der Corput's work on the stationary phase method [76].

Lemma 3.6.2. *Let λ be a positive real. Suppose that $\phi : (a, b) \to \mathbb{R}$ is a smooth function in (a, b), and suppose that $\left|\phi^{(k)}(x)\right| \geq 1$ for all $x \in (a, b)$. Then,*

$$\left| \int_a^b e^{i\lambda\phi(x)}\mathrm{d}x \right| \leq c_k\lambda^{-1/k}$$

holds when:

(i) $k \geq 2$, *or*

(ii) $k = 1$ *and* $\phi'(x)$ *is monotonic.*

The bound c_k *is independent of* ϕ *and* λ.

Proof. See [73, Proposition 2, p.332]. □

Proof of Lemma 3.6.1. The estimates concerning $\upsilon_{g,j}$ and $\upsilon_{h,k}$ can be easily established by using integration by parts. As an alternative approach, one may use Lemma 3.6.2 as below. Now we come to prove the claimed estimate about the function $\upsilon_{f,i}$.

For $\boldsymbol{\beta} = (\beta_d, \beta_\theta) \in \mathbb{R}^2$ we put

$$\upsilon_f(\boldsymbol{\beta}) = \int_{1/2}^{2} e(\beta_d \gamma^d + \beta_\theta \gamma^\theta) \mathrm{d}\gamma.$$

It is enough to prove that

$$\upsilon_f(\boldsymbol{\beta}) \ll \frac{1}{(1 + |\beta_d| + |\beta_\theta|)^{1/\theta}}. \tag{3.6.4}$$

The desired estimate for the function $\upsilon_{f,i}$ follows by a change of variables replacing γ by $x_i^\star P\gamma$. Then one may apply (3.6.4) with $a_i(x_i^\star P)^d \beta_d$ in place of β_d and $\lambda_i(x_i^\star P)^\theta \beta_\theta$ in place of β_θ.

It is apparent that $|\upsilon_f(\boldsymbol{\beta})| \leq 3/2 \ll 1$. So, if $|\beta_d| + |\beta_\theta| < 1$ then (3.6.4) trivially holds. Hence, in the rest of the proof we may suppose that $|\beta_d| + |\beta_\theta| \geq 1$. For $\gamma \in [1, 2]$ we define the function

$$\phi(\gamma) = \beta_d \gamma^d + \beta_\theta \gamma^\theta.$$

We distinguish the following two cases about β_d and β_θ.

Case (1). Suppose that $|\beta_\theta| > |\beta_d|$. Recall that d is a positive integer such that $\theta > d + 1$. This last condition implies that $d < \lfloor \theta \rfloor$. Temporarily we write $n = \lfloor \theta \rfloor$. Hence, for $\gamma \in [1/2, 2]$ one has

$$|\phi^{(n)}(\gamma)| = C_n |\beta_\theta| \gamma^{\theta - n} \geq C_n \left(\frac{1}{2}\right)^{\theta - n} |\beta_\theta|,$$

where we put $C_n = \theta(\theta - 1) \cdots (\theta - n + 1)$. Put $C = C_n \left(\frac{1}{2}\right)^{\theta - n}$. One may now take $\lambda = C|\beta_\theta|$ and apply Lemma 3.6.2 with $k = n$ to the function

$$\gamma \mapsto \frac{1}{C|\beta_\theta|} \phi(\gamma).$$

Since $|\beta_\theta| > |\beta_d|$ and $|\beta_d| + |\beta_\theta| \geq 1$, we deduce that

$$\int_{1/2}^{2} e(\phi(\gamma)) \mathrm{d}\gamma \leq C^{-1/n} |\beta_\theta|^{-1/n} \ll \frac{1}{(1 + |\beta_d| + |\beta_\theta|)^{1/\theta}},$$

which confirms (3.6.4).

Case (2). Suppose that $|\beta_\theta| \leq |\beta_d|$. One has

$$|\phi^{(d)}(\gamma)| = \left|d!\beta_d + C_d\beta_\theta\gamma^{\theta-d}\right|,$$

where we put $C_d = \theta(\theta-1)\cdots(\theta-d+1)$. In order to give a lower bound for the quantity $\left|\phi^{(d)}(\gamma)\right|$ we examine separately the following two scenarios.

Suppose that

$$\frac{1}{2}d!|\beta_d| \geq C_d 2^{\theta-d}|\beta_\theta|.$$

By the triangle triangle inequality one may infer for $\gamma \in [1/2, 2]$ that

$$|\phi^{(d)}(\gamma)| > d!|\beta_d| - C_d\gamma^{\theta-d}\,|\beta_\theta| \geq d!|\beta_d| - C_d 2^{\theta-d}\,|\beta_\theta| \geq \frac{1}{2}d!|\beta_d|.$$

One may now take $\lambda = 2^{-1}d!|\beta_d|$ and apply Lemma 3.6.2 with $k = d$ to the function

$$\gamma \mapsto \frac{1}{2^{-1}d!|\beta_d|}\phi(\gamma).$$

Since $|\beta_d| \geq |\beta_\theta|$ and $|\beta_d| + |\beta_\theta| \geq 1$, we deduce that

$$\int_{1/2}^{2} e(\phi(\gamma))\mathrm{d}\gamma \leq (2^{-1}d!)^{-1/d}\,|\beta_d|^{-1/d} \ll \frac{1}{(1+|\beta_d|+|\beta_\theta|)^{1/\theta}},$$

which again confirms (3.6.4).

Next, we suppose that

$$\frac{1}{2}d!|\beta_d| < C_d 2^{\theta-d}|\beta_\theta|.$$

Since we assume as well that $|\beta_d| \geq |\beta_\theta|$ one may now suppose that $|\beta_d| \asymp |\beta_\theta|$. In such a situation an application of Lemma 3.6.2 with $k = n$ as in *Case* (1) yields

$$\int_{1/2}^{2} e\left(\phi(\gamma)\right)\mathrm{d}\gamma \ll |\beta_\theta|^{-1/\theta} \ll \frac{1}{(1+|\beta_d|+|\beta_\theta|)^{1/\theta}},$$

and thus the proof is now complete. □

Define $\Delta = \Delta(\theta, d, \ell, m, n) > 0$ via

$$\Delta(\theta, d, \ell, m, n) = \min\left\{\frac{m}{\theta} + \frac{\ell}{2\theta} - 1,\ \frac{n}{d} + \frac{\ell}{2\theta} - 1\right\}. \tag{3.6.5}$$

Note that the assumptions $\ell + m \geq A_\theta + 1$ and $\ell \geq \max\{\lceil 2\theta(1-n/d)\rceil,\ 1\}$ ensure that $\Delta > 0$.

Lemma 3.6.3. *One has*

$$\mathfrak{J}^{\pm}(P^\xi, P^{\delta_0}) = 2\tau\mathfrak{J}(\infty) + o\left(P^{s-(\theta+d)}\right).$$

Proof. For $|\beta_\theta| < P^{-\theta+\delta_0}$ by (3.2.4) one has that

$$\mathfrak{J}^{\pm}(P^\xi, P^{\delta_0}) = \left(2\tau + O\left((\log P)^{-2}\right)\right)\mathfrak{J}(P^\xi, P^{\delta_0}). \tag{3.6.6}$$

By Lemma 3.6.1 and a trivial estimate one has that

$$V(\boldsymbol{\beta}) \ll P^s \left(1 + |\beta_\theta| P^\theta\right)^{-m/\theta} \left(1 + |\beta_d| P^d\right)^{-n/d} \left(1 + |\beta_d| P^d + |\beta_\theta| P^\theta\right)^{-\ell/\theta}.$$

Using the trivial estimate

$$\alpha^{1/2}\beta^{1/2} \leq \max\{\alpha, \beta\} \ll 1 + \alpha + \beta,$$

the preceding inequality now yields

$$\begin{aligned} V(\boldsymbol{\beta}) &\ll P^s \left(1 + |\beta_\theta| P^\theta\right)^{-m/\theta - \ell/2\theta} \left(1 + |\beta_d| P^d\right)^{-n/d - \ell/2\theta} \\ &\ll P^s \left(1 + |\beta_\theta| P^\theta\right)^{-(1+\Delta)} \left(1 + |\beta_d| P^d\right)^{-(1+\Delta)}. \end{aligned} \tag{3.6.7}$$

Temporarily we write $\mathcal{B}$ to denote the box $[P^{-\theta+\xi}, P^{-\theta+\xi}] \times [-P^{-d+\delta_0}, P^{-d+\delta_0}]$. If $\boldsymbol{\beta} \in \mathbb{R}^2 \setminus \mathcal{B}$ then we either have $|\beta_\theta| P^\theta \geq P^{\delta_0}$ or $|\beta_d| P^d \geq P^\xi$. By the preceding estimate we may infer that

$$\begin{aligned} \mathfrak{J}(P^\xi, P^{\delta_0}) - \mathfrak{J}(\infty) &\ll P^s \left(\int_{|\beta_\theta| P^\theta \geq P^{\delta_0}} \int_{-\infty}^{\infty} V(\boldsymbol{\beta}) \mathrm{d}\boldsymbol{\beta} + \int_{-\infty}^{\infty} \int_{|\beta_d| P^d \geq P^\xi} V(\boldsymbol{\beta}) \mathrm{d}\boldsymbol{\beta} \right) \\ &\ll P^{s-(\theta+d)-\Delta\delta_0} + P^{s-(\theta+d)-\Delta\xi} \\ &\ll o\left(P^{s-(\theta+d)}\right). \end{aligned}$$

Therefore, by (3.6.6) we deduce that

$$\mathfrak{J}^{\pm}(P^\xi, P^{\delta_0}) = 2\tau\mathfrak{J}(\infty) + o\left(P^{s-(\theta+d)}\right),$$

which is what we wanted to prove. □

After these preliminary results we now come to the heart of the singular integral analysis. The approach we take for studying the singular integral $\mathfrak{J}$ is essentially the treatment of Schmidt as presented in [68]. The validity of the results below should come with no surprise to the experts and to those who are familiar with the paper of Schmidt. For the sake of completeness we have decided to include the proofs that are related to the system under investigation. This is mainly due to the nature of the system (3.1.2), which consists of an equation and an inequality of fractional degree.

One can plainly extend the definition of $\mathfrak{F}$ and $\mathfrak{D}$ given in (3.1.2) to s tuples by taking the additional coefficients to be equal to zero. Namely, for an s tuple $\mathbf{x}$ we can rewrite $\mathfrak{F}$ and $\mathfrak{D}$ equivalently in the shape

$$\begin{cases} \mathfrak{F}(\mathbf{x}) = \lambda_1 x_1^\theta + \cdots + \lambda_\ell x_\ell^\theta + \mu_1 x_{\ell+1}^\theta + \cdots + \mu_{\ell+m} x_{\ell+m}^\theta + 0 x_{\ell+m+1}^\theta + \cdots + 0 x_s^\theta \\ \mathfrak{D}(\mathbf{x}) = a_1 x_1^d + \cdots + a_\ell x_\ell^d + 0 x_{\ell+1}^d + \cdots + 0 x_{s-n}^d + b_1 x_{s-n+1}^d + \cdots + b_n x_s^d. \end{cases} \tag{3.6.8}$$

So, from now on we take the argument in the expressions $\mathfrak{F}$ and $\mathfrak{D}$ to be s tuples. For convenience in the following, we write $\mathcal{B}$ to denote the box defined by

$$\mathcal{B} = \left[\frac{1}{2}\eta_1, 2\eta_1\right] \times \cdots \times \left[\frac{1}{2}\eta_s, 2\eta_s\right],$$

where $\boldsymbol{\eta} = (\mathbf{x}^\star, \mathbf{y}^\star, \mathbf{z}^\star)$ is a non-singular real solution of the system (3.1.3), with $\mathfrak{F}$ and $\mathfrak{D}$ defined as in (3.6.8). Note that with this notation, we count solutions to the system (3.1.2) with $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in P\mathcal{B}$.

We define the integral

$$\mathcal{K}(\boldsymbol{\beta}) = \int_{\mathcal{B}} e\left(\beta_\theta \mathfrak{F}(\boldsymbol{\gamma}) + \beta_d \mathfrak{D}(\boldsymbol{\gamma})\right) \mathrm{d}\boldsymbol{\gamma}.$$

For future reference we note here that by (3.6.7) with $P = 1$ and since $\operatorname{meas}(\mathfrak{B}) = O(1)$ one has

$$\mathcal{K}(\boldsymbol{\beta}) \ll (1 + |\beta_\theta|)^{-(1+\Delta)} (1 + |\beta_d|)^{-(1+\Delta)}. \tag{3.6.9}$$

Moreover, we set

$$\mathfrak{J}_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{K}(\boldsymbol{\beta}) \mathrm{d}\boldsymbol{\beta}. \tag{3.6.10}$$

In the light of (3.6.9) the integral $\mathfrak{J}_0$ is well-defined and absolutely convergent. One may express the complete singular integral $\mathfrak{J}(\infty)$ in terms of $\mathfrak{J}_0$. Replace γ by γP in (3.6.1). Then make a change of variables in the right hand side of (3.6.3) by putting

$$\begin{pmatrix} \beta_\theta \\ \beta_d \end{pmatrix} = \begin{pmatrix} P^{-\theta} & 0 \\ 0 & P^{-d} \end{pmatrix} \begin{pmatrix} \beta'_\theta \\ \beta'_d \end{pmatrix}.$$

This yields

$$\mathfrak{J}(\infty) = P^{s-(\theta+d)} \mathfrak{J}_0. \tag{3.6.11}$$

We may now focus in analysing the integral $\mathfrak{J}_0$. To do so, we make use of a family of approximate singular integrals. For $T \geq 1$ we put

$$\mathfrak{J}(T) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{K}(\boldsymbol{\beta}) k_T(\boldsymbol{\beta}) \mathrm{d}\boldsymbol{\beta}, \tag{3.6.12}$$

where

$$k_T(\boldsymbol{\beta}) = \left(\frac{\sin(\pi\beta_\theta/T)}{\pi\beta_\theta/T}\right)^2 \left(\frac{\sin(\pi\beta_d/T)}{\pi\beta_d/T}\right)^2.$$

Note again that by (3.6.9) the integral $\mathfrak{J}(T)$ is well-defined and absolutely convergent. Two are the key properties of the family of integrals $\mathfrak{J}(T)$. Firstly that $\mathfrak{J}(T) \gg 1$ and secondly that as $T \to \infty$ one has $\mathfrak{J}(T) \to \mathfrak{J}_0$. To begin with, let us rewrite the integrals $\mathfrak{J}(T)$ using a Fourier transform formula. For $T \geq 1$ we put

$$\psi_T(y) = \begin{cases} T\left(1 - T|y|\right), & \text{when } |y| \leq T^{-1}, \\ 0, & \text{when } |y| > T^{-1}. \end{cases} \tag{3.6.13}$$

A standard calculation as presented for example in [29, Lemma 20.1] reveals that

$$\psi_T(y) = \int_{-\infty}^{\infty} e(\beta y) \left(\frac{\sin(\pi\beta/T)}{\pi\beta/T} \right)^2 \mathrm{d}\beta,$$

where clearly the integral is absolutely convergent. One may rewrite the integral $\mathfrak{J}(T)$ defined in (3.6.12) as follows

$$\mathfrak{J}(T) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{\mathcal{B}} e(\beta_\theta \mathfrak{F}(\boldsymbol{\gamma}) + \beta_d \mathfrak{D}(\boldsymbol{\gamma}) \mathrm{d}\boldsymbol{\gamma} \right) k_T(\boldsymbol{\beta}) \mathrm{d}\boldsymbol{\beta}.$$

Hence, invoking Fubini's theorem and appealing to (3.6.13) one has

$$\mathfrak{J}(T) = \int_{\mathcal{B}} \psi_T(\mathfrak{F}(\boldsymbol{\gamma}))\psi_T(\mathfrak{D}(\boldsymbol{\gamma}))\mathrm{d}\boldsymbol{\gamma}. \tag{3.6.14}$$

At this point we pause for a moment in order to exploit the assumption we have made that the system (3.1.2) satisfies the local solubility condition. The conclusion we establish below plays an essential role in demonstrating that $\mathfrak{J}(T) \gg 1$. The proof proceeds as in [83, Lemma 6.2], namely by using the implicit function theorem. So, the fact that $\mathfrak{F}$ is a generalised polynomial of fractional degree θ does not affect things. We include a proof for the sake of completeness.

Lemma 3.6.4. *There exists locally an $(s-2)$-dimensional subspace $\mathcal{U}$ of positive $(s-2)$-volume in a neighbourhood of $\boldsymbol{\eta}$, on which one has $\mathfrak{F} = \mathfrak{D} = 0$. In particular, there exists a real solution $\boldsymbol{\eta}'$ to the system (3.1.3), with $\eta_i' \neq 0$ for all i.*

Proof. Recall that $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s)$ is a non-singular real solution of the system (3.1.2) with $\mathfrak{F}, \mathfrak{D}$ as in (3.6.8). By relabelling if necessary the variables, one has

$$\det \begin{pmatrix} \theta\lambda_1\eta_1^{\theta-1} & \theta\lambda_2\eta_2^{\theta-1} \\ da_1\eta_1^{d-1} & da_2\eta_2^{d-1} \end{pmatrix} = \theta d\eta_1\eta_2 \left(\lambda_1 a_2 \eta_1^{\theta-2}\eta_2^{d-2} - \lambda_2 a_2 \eta_1^{d-2}\eta_2^{d-2} \right) \neq 0.$$

Hence, we deduce that $\eta_1, \eta_2 \neq 0$. Let $\mathcal{A} \subset \mathbb{R}^{2+(s-2)}$ be a neighbourhood of the point $\boldsymbol{\eta}$. Consider the map

$$\Phi : \mathcal{A} \to \mathbb{R}^2, \qquad \mathbf{x} \mapsto \Phi(\mathbf{x}) = (\mathfrak{F}(\mathbf{x}), \mathfrak{D}(\mathbf{x})).$$

We know that $\Phi(\boldsymbol{\eta}) = 0$. By the implicit function theorem we know that there exists a neighbourhood $\mathcal{V} \subset \mathbb{R}^{s-2}$ around the point $(\eta_3, \ldots, \eta_s)$ and a unique continuously differentiable map $\mathbf{g} : \mathcal{V} \to \mathbb{R}^2$ such that for all $\boldsymbol{\zeta} = (\zeta_3, \ldots, \zeta_s) \in \mathcal{V}$ one has

$$\begin{cases} \mathfrak{F}(\mathbf{g}(\boldsymbol{\zeta}), \boldsymbol{\zeta}) = 0 \\ \mathfrak{D}(\mathbf{g}(\boldsymbol{\zeta}), \boldsymbol{\zeta}) = 0. \end{cases} \tag{3.6.15}$$

Thus, we have showed the existence of an $(s-2)$-dimensional subspace in the neighbourhood of $(\eta_3, \ldots, \eta_s)$, which has positive $(s-2)$-volume and on which one has $\mathfrak{F} = \mathfrak{D} = 0$. We denote this subspace by $\mathcal{U}$. This establishes the main part in the statement of the lemma.

For the second assertion we argue as follows. One can choose $\zeta_i \in \mathcal{V}$ sufficiently close to η_i for $3 \leq i \leq s$. Namely, choose ζ_i such that $|\zeta_i - \eta_i|$ is sufficiently small. Then, we can solve the system (3.6.15) with respect to $\mathbf{g} =: (\zeta_1, \zeta_2)$. Hence, we have found a tuple $\boldsymbol{\eta}' = (\zeta_1, \zeta_2, \boldsymbol{\zeta})$ which satisfies $\mathfrak{F}(\boldsymbol{\eta}') = \mathfrak{D}(\boldsymbol{\eta}') = 0$. Recall, that $\eta_1, \eta_2 \neq 0$. Hence, by continuity we obtain that $\zeta_1, \zeta_2 \neq 0$. Therefore, we can conclude that $\zeta_i \neq 0$ for $3 \leq i \leq s$. Thus, we have shown the existence of a real solution $\boldsymbol{\eta}'$ with all of its components being non-zero. $\square$

We now exploit the conclusion of Lemma 3.6.4, in order to prove that $\mathfrak{J}(T) \gg 1$. Here we follow [68, Lemma 2].

Lemma 3.6.5. *One has*

$$\mathfrak{J}(T) \gg 1.$$

Proof. With the notation as in Lemma 3.6.4, we write $\boldsymbol{\eta}' = (\zeta_1, \zeta_2, \boldsymbol{\zeta})$ to denote a real solution to the system (3.6.8) with $\eta_i' \neq 0$ for $1 \leq i \leq s$. We put $\boldsymbol{\zeta} = (\zeta_3, \ldots, \zeta_s)$. Note here that we assume $\zeta_i \neq 0$ for $3 \leq i \leq s$. For $\epsilon > 0$ we define

$$S_\epsilon = \{(\boldsymbol{\xi}, \boldsymbol{\zeta}) : \boldsymbol{\zeta} \in \mathcal{U} \text{ such that } \|\mathbf{g}(\boldsymbol{\zeta}) - \boldsymbol{\xi}\|_2 < \epsilon\},$$

where $\|\cdot\|_2$ stands for the usual euclidean norm in $\mathbb{R}^{s-2}$. In the set S_ϵ we consider points $\boldsymbol{\xi} \in \mathbb{R}^2$ which belong to a neighbourhood of the point $\mathbf{g}(\boldsymbol{\zeta})$. Since $\mathcal{U}$ is a subset of the interior of the box $\mathcal{B}$, one may now consider sufficiently small ϵ so that $S_\epsilon \subset \mathcal{B}$. Moreover, by Lemma 3.6.4 we know that $\mathbf{g}(\boldsymbol{\zeta}) \neq \mathbf{0}$. Hence, it becomes apparent that the set S_ϵ has a positive s-volume.

When viewed as real valued functions in s variables, the generalised polynomial $\mathfrak{F}$ and the polynomial $\mathfrak{D}$ are continuously differentiable in the box $\mathcal{B}$, which is a compact subset of $\mathbb{R}^s$. Hence, $\mathfrak{F}$ and $\mathfrak{D}$ satisfy the Lipschitz condition with some constants K_1 and K_2 respectively. Put

$$c = \frac{1}{2\max\{K_1, K_2\}} > 0.$$

From now on we take T sufficiently large so that $S_{cT^{-1}} \subset \mathcal{B}$.

For $(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in S_{cT^{-1}}$ one has

$$\frac{|\mathfrak{D}(\boldsymbol{\xi}, \boldsymbol{\zeta}) - \mathfrak{D}(\mathbf{g}(\boldsymbol{\zeta}), \boldsymbol{\zeta})|}{\|(\boldsymbol{\xi}, \boldsymbol{\zeta}) - (\mathbf{g}(\boldsymbol{\zeta}), \boldsymbol{\zeta})\|_2} < K_2.$$

By (3.6.15) one has $\mathfrak{D}(\mathbf{g}(\boldsymbol{\zeta}), \boldsymbol{\zeta}) = 0$. Moreover, one has $\|(\boldsymbol{\xi} - \mathbf{g}(\boldsymbol{\zeta}), \mathbf{0})\|_2 < c/T$ and so the above inequality yields

$$|\mathfrak{D}(\boldsymbol{\xi}, \boldsymbol{\zeta})| < \frac{c}{T} K_2 < \frac{1}{2T}.$$

Thus, for $\boldsymbol{\gamma} = (\boldsymbol{\xi}, \boldsymbol{\zeta}) \in S_{cT^{-1}}$ we deduce

$$\psi_T(\mathfrak{D}(\boldsymbol{\gamma})) = \max\{0, T(1 - T|\mathfrak{D}(\boldsymbol{\gamma})|)\} \geq \frac{T}{2}.$$

Similarly, when $(\boldsymbol{\xi}, \boldsymbol{\zeta}) \in S_{cT^{-1}}$ one may prove that

$$|\mathfrak{F}(\boldsymbol{\xi}, \boldsymbol{\zeta})| < \frac{1}{2T}.$$

Thus, we may again deduce that for $\boldsymbol{\gamma} = (\boldsymbol{\xi}, \boldsymbol{\zeta}) \in S_{cT^{-1}}$ one has

$$\psi_T(\mathfrak{F}(\boldsymbol{\gamma})) = \max\{0, T(1 - T|\mathfrak{F}(\boldsymbol{\gamma})|)\} \geq \frac{T}{2}.$$

Note now that the set $S_{cT^{-1}}$ has positive s-volume which satisfies $\gg T^{-2}$. Hence, from the above conclusions and (3.6.14) one has

$$\mathfrak{J}(T) = \int_{\mathcal{B}} \psi_T(\mathfrak{D}(\boldsymbol{\gamma}))\psi_T(\mathfrak{F}(\boldsymbol{\gamma}))\mathrm{d}\boldsymbol{\gamma} \gg \int_{S_{cT^{-1}}} \left(\frac{T}{2}\right)^2 \gg \frac{1}{4},$$

which completes the proof of the lemma. □

Next, we establish the second key property of the family of approximate integral $\mathfrak{J}(T)$.

Lemma 3.6.6. *One has*

$$\mathfrak{J}(T) = \mathfrak{J}_0 + O\left(T^{-\Delta}\right),$$

where $\Delta > 0$ is defined in (3.6.5). In particular, the limit of $\mathfrak{J}(T)$ as $T \to \infty$ exists and equals to $\mathfrak{J}_0$.

Proof. By (3.6.9) and (3.6.12) we infer that

$$\begin{aligned}\mathfrak{J}_0 - \mathfrak{J}(T) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{K}(\boldsymbol{\beta})\left(1 - k_T(\boldsymbol{\beta})\right)\mathrm{d}\boldsymbol{\beta} \\ &\ll \int_0^{\infty}\int_0^{\infty} (1+\beta_\theta)^{-(1+\Delta)}(1+\beta_d)^{-(1+\Delta)}\left(1 - k_T(\boldsymbol{\beta})\right)\mathrm{d}\boldsymbol{\beta}.\end{aligned}$$

Let $\beta \in \mathbb{R}$ and let T be large enough so that $\frac{\pi|\beta|}{T} < 1$. Then one has

$$\left(\frac{\sin(\pi\beta/T)}{\pi\beta/T}\right)^2 = 1 + O\left(\frac{|\beta|^2}{T^2}\right),$$

which yields that

$$1 - \left(\frac{\sin(\pi\beta/T)}{\pi\beta/T}\right)^2 \ll \min\left\{1, \frac{|\beta|^2}{T^2}\right\} \ll \frac{|\beta|^2}{T^2 + |\beta|^2},$$

and thus we deduce that

$$1 - k_T(\boldsymbol{\beta}) \ll \frac{|\beta_\theta|^2}{T^2 + |\beta_\theta|^2} + \frac{|\beta_d|^2}{T^2 + |\beta_d|^2}.$$

We may now finish the proof easily. By symmetry one has

$$\begin{aligned}\mathfrak{J}_0 - \mathfrak{J}(T) &\ll \left(\int_0^{\infty}(1+\beta_\theta)^{-(1+\Delta)}\mathrm{d}\beta_\theta\right)\left(\int_0^{\infty}(1+\beta_d)^{-(1+\Delta)}\frac{|\beta_d|^2}{T^2+|\beta_d|^2}\mathrm{d}\beta_d\right) \\ &\ll T^{-2}\int_0^T \beta_d^{1-\Delta}\mathrm{d}\beta_d + \int_T^{\infty}\beta_d^{-(1+\Delta)}\mathrm{d}\beta_d \\ &\ll T^{-\Delta},\end{aligned}$$

which completes the proof. □

Below we put together the outcomes of the so far analysis, in order to deduce the desired estimate for the truncated singular integral defined in (3.6.2).

Lemma 3.6.7. *One has*

$$\mathfrak{J}^{\pm}(P^{\xi}, P^{\delta_0}) = 2\tau\mathfrak{J}_0 P^{s-(\theta+d)} + o\left(P^{s-(\theta+d)}\right),$$

where $\mathfrak{J}_0 > 0$ is defined in (3.6.10).

Proof. Combining Lemma 3.6.3 and relation (3.6.11) we deduce that

$$\mathfrak{J}^{\pm}(P^{\xi}, P^{\delta_0}) = 2\tau\mathfrak{J}_0 P^{s-(\theta+d)} + o\left(P^{s-(\theta+d)}\right).$$

Moreover, by Lemma 3.6.5 and Lemma 3.6.6 we infer that $\mathfrak{J}_0 \gg 1$ which completes the proof. □

3.6.2 Singular series analysis

We now study the singular series related to the equation $\mathfrak{D}(\mathbf{x}, \mathbf{z}) = 0$. For $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ we write

$$S(q, a) = \sum_{z=1}^{q} e\left(\frac{az^d}{q}\right).$$

Furthermore we put

$$T(q, a) = q^{-(\ell+n)} \prod_{i=1}^{\ell} S(q, aa_i) \prod_{k=1}^{n} S(q, ab_k).$$

Next, we introduce the truncated singular series and its completed analogue

$$\mathfrak{S}(P^{\xi}) = \sum_{1\leq q\leq P^{\xi}} \sum_{\substack{a=1\\(a,q)=1}}^{q} T(q, a) \qquad \text{and} \qquad \mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} T(q, a).$$

Lemma 3.6.8. *Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$. Then for each index i and k one has*

$$S(q, aa_i),\ \ S(q, ab_k) \ll q^{1-1/d}.$$

Proof. By [29, Lemma 6.4] we know that when $(a, q) = 1$ one has

$$S(q, a) \ll q^{1-1/d}.$$

Fix an index i. Note that one has

$$S(q, aa_i) = \sum_{z=1}^{q} e\left(\frac{a_i az^d}{q}\right) = (q, a_i) S\left(\frac{q}{(q, a_i)}, \frac{a_i a}{(q, a_i)}\right).$$

Since $(a,q)=1$ one has $\left(\frac{q}{(q,a_i)},\frac{a_ia}{(q,a_i)}\right)=1$. Thus we derive that

$$S\left(\frac{q}{(q,a_i)},\frac{a_ia}{(q,a_i)}\right)\ll q^{1-1/d},$$

which in turn, and since a_i is a fixed integer, delivers the estimate

$$S(q,aa_i)\ll q^{1-1/d}.$$

Similarly we argue for $S(q,ab_k)$. □

Lemma 3.6.9. *Provided that $\ell+n\geq A_d+1$ the singular series is absolutely convergent. Moreover one has $\mathfrak{S}>0$ and*

$$\mathfrak{S}(P^{\xi})=\mathfrak{S}+O\left(P^{-\xi/d}\right).$$

Proof. The first two claims follow from the analysis of Davenport as presented in [29, Sections 5 & 6]. Recall that we write $A_d=d^2$. By [31, Theorem 1] we know that if $\ell+n\geq A_d+1$ then the singular series is absolutely convergent and positive. For the last assertion note that by Lemma 3.6.8 one has

$$\left|\mathfrak{S}-\mathfrak{S}(P^{\xi})\right|\leq\sum_{q>P^{\xi}}\sum_{\substack{a=1\\(a,q)=1}}^{q}|T(q,a)|\ll\sum_{q>P^{\xi}}q^{1-(\ell+n)/d}\ll P^{(2-(\ell+n)/d)\xi}.$$

For $d\geq 2$ one has $\ell+n\geq A_d+1\geq 2d+1$, where in the second inequality, the equality case holds only when $d=2$. Thus, we obtain $\frac{\xi}{d}\leq\left(\frac{\ell+n}{d}-2\right)\xi$. The previous estimate now delivers

$$\left|\mathfrak{S}-\mathfrak{S}(P^{\xi})\right|\ll P^{-\xi/d},$$

which completes the proof. □

3.7 The asymptotic formula

We now combine the results from the previous two sections to establish the anticipated asymptotic formula for the counting function $\mathcal{N}(P)$.

For $\alpha_d\in\mathfrak{N}_{\xi}(q,a)$ we write $\alpha_d=\beta_d+a/q$ with $|\beta_d|<P^{-d+\xi}$. From now on we take $\boldsymbol{\beta}=(\beta_d,\alpha_\theta)$, with $\alpha_\theta\in\mathfrak{M}$. For each i,j and k we define the approximate generating functions

$$f_i^{\star}(\boldsymbol{\beta})=\frac{1}{q}S_{f,i}(q,a)v_{f,i}(\boldsymbol{\beta}),\quad g_j^{\star}(\boldsymbol{\beta})=v_{g,j}(\boldsymbol{\beta}),\quad h_k^{\star}(\boldsymbol{\beta})=\frac{1}{q}S_{h,k}(q,a)v_{h,k}(\boldsymbol{\beta}).$$

Put

$$\mathcal{F}^{\star}(\boldsymbol{\beta})=\prod_{i=1}^{\ell}f_i^{\star}(\boldsymbol{\beta})\prod_{j=1}^{m}g_j^{\star}(\boldsymbol{\beta})\prod_{k=1}^{n}h_k^{\star}(\boldsymbol{\beta}).$$

We wish to compare $\mathcal{F}(\boldsymbol{\alpha})$ with $\mathcal{F}^{\star}(\boldsymbol{\beta})$. Below we record a consequence of Poisson's summation formula.

Lemma 3.7.1. *Let $f : [a, b] \to \mathbb{R}$ be a function differentiable in $[a, b]$. Suppose that $f'(x)$ is monotonic, and suppose that $|f'(x)| \le A < 1$ for all $x \in [a, b]$. Then*

$$\sum_{a<x\le b} e\left(f(x)\right) = \int_a^b e\left(f(x)\right) \mathrm{d}x + O(1).$$

Proof. See [74, Lemma 4.8]. □

Lemma 3.7.2. *For each index i, j, k and for any $\boldsymbol{\alpha} = (\alpha_d, \alpha_\theta) \in \mathfrak{N}_\xi(q, a) \times \mathfrak{M}$ one has*

(i) $f_i(\boldsymbol{\alpha}) - f_i^\star(\boldsymbol{\beta}) \ll P^{\delta_0+\xi}$ *;*

(ii) $g_j(\boldsymbol{\alpha}) - g_j^\star(\boldsymbol{\beta}) \ll 1$*;*

(iii) $h_k(\boldsymbol{\alpha}) - h_k^\star(\boldsymbol{\beta}) \ll P^{2\xi}$.

Proof. For the estimate (iii) one may argue as in [29, Lemma 4.2].

For the estimate (ii) one may apply Lemma 3.7.1. Fix an index j. Recall from (3.6.1) the definition of the function $\upsilon_{g,j}(\boldsymbol{\beta})$. Then, the claimed estimate reads

$$\sum_{\frac{1}{2}y_i^\star P<y\le 2y_i^\star P} e(\mu_j\alpha_\theta y^\theta) - \int_{\frac{1}{2}y_j^\star P}^{2y_j^\star P} e(\mu_j\alpha_\theta\gamma^\theta)\mathrm{d}\gamma = O(1).$$

By taking the complex conjugate it suffices to prove the above estimate when $\alpha_\theta > 0$. For a real variable t we define the function

$$\phi : \left(\frac{1}{2}y_i^\star P, 2y_i^\star P\right] \to \mathbb{R}, \qquad \phi(t) = \mu_j\alpha_\theta t^\theta.$$

The function $\phi''(t)$ is of fixed sign and so $\phi'(t)$ is monotonic. Moreover, for $\alpha_\theta \in \mathfrak{M}$ and for large enough P one has

$$|\phi'(t)| = |\mu_j|\theta\alpha_\theta t^{\theta-1} \le |\mu_j|\theta(2y_i^\star)^{\theta-1}P^{-1+\delta_0} < 1,$$

where recall from (3.2.7) that $\delta_0 < 1$. Thus, Lemma 3.7.1 is applicable and yields the desired conclusion.

Now we prove estimate (i). Here we argue as in [29, Lemma 4.2]. We fix an index i. Decomposing into residue classes modulo q and writing $x = qy + z$ with $1 \le z \le q$ we obtain

$$\begin{aligned} f_i(\boldsymbol{\alpha}) &= \sum_{z=1}^{q}\sum_{y\in I(z)} e\left(a_i(\beta_d + a/q)(qy+z)^d + \lambda_i\alpha_\theta(qy+z)^\theta\right) \\ &= \sum_{z=1}^{q} e\left(a_i a z^d/q\right)\sum_{y\in I} e\left(a_i\beta_d(qy+z)^d + \lambda_i\alpha_\theta(qy+z)^\theta\right), \end{aligned} \tag{3.7.1}$$

where $I = I(z)$ is the interval defined by

$$I(z) = \left(\frac{\frac{1}{2}x_i^\star P - z}{q}, \ \frac{2x_i^\star P - z}{q}\right].$$

For ease of notation we denote the endpoints of the interval I by A and B. So $I = (A, B]$.

For $t \in \mathbb{R}$ we put
$$\phi_i(t) = e\left(a_i\beta_d(qt+z)^d + \lambda_i\alpha_\theta(qt+z)^\theta\right).$$

The function ϕ_i is a holomorphic complex valued function of the real variable t. Consider an arbitrary interval $[x, x+1] \subset \mathbb{R}$ of length equal to 1. By the fundamental theorem of calculus one has for any $t \in [x, x+1]$ that

$$|\phi_i(t) - \phi_i(x)| = \left|\int_x^t \phi_i'(u)\mathrm{d}u\right| \leq \max_{u\in[x,x+1]} |\phi_i'(u)|.$$

One may break the interval I into $\ll B - A = O\left(Pq^{-1}\right)$ unit intervals of the shape $[x, x+1]$ with $x \in \mathbb{Z}$, together with two possible broken intervals in the case where at least one of the endpoints A and B of the interval I is not an integer. Then, we deduce that

$$\begin{aligned}\left|\sum_{A<y\leq B}\phi_i(y) - \int_A^B \phi_i(t)\mathrm{d}t\right| &\ll \sum_{A<y\leq B}\int_y^{y+1} |\phi_i(y)-\phi_i(t)|\,\mathrm{d}t + \max_{A<y\leq B}|\phi_i(t)| \\ &\ll Pq^{-1}\max_{A<t\leq B}|\phi_i'(t)| + \max_{A<t\leq B}|\phi_i(t)|.\end{aligned}$$

Clearly, $|\phi_i(t)| \leq 1$ for all t. One has

$$\phi_i'(t) = 2\pi i\left(a_i dq\beta_d(qt+z)^{d-1} + \lambda_i\theta q\alpha_\theta(qt+z)^{\theta-1}\right)\phi_i(t).$$

Hence, for any $t \in I$ one has

$$|\phi_i'(t)| \ll q|\beta_d|P^{d-1} + q|\alpha_\theta|P^{\theta-1}.$$

Therefore for $(\alpha_d, \alpha_\theta) \in \mathfrak{N}_\xi(q,a) \times \mathfrak{M}$ and since $\xi < \delta_0$, the preceding estimate now delivers

$$\left|\sum_{A<y\leq B}\phi_i(y) - \int_A^B \phi_i(t)\mathrm{d}t\right| \ll P^{\delta_0}. \tag{3.7.2}$$

We put $qt + z = \gamma$ and make a change of variables. Then one has

$$\int_A^B \phi_i(t)\mathrm{d}t = \frac{1}{q}\int_{\frac{1}{2}x_i^\star P}^{2x_i^\star P} e(a_i\beta_d\gamma^d + \lambda_i\alpha_\theta\gamma^\theta)\mathrm{d}\gamma = \frac{1}{q}\upsilon_{f,i}(\boldsymbol{\beta}),$$

where bear in mind that $\boldsymbol{\beta} = (\beta_d, \alpha_\theta) = (\alpha_d - a/q, \alpha_\theta)$. Putting together (3.7.1) and (3.7.2) yields

$$\begin{aligned} f_i(\boldsymbol{\alpha}) &= \sum_{z=1}^{q} e\left(a_i a z^d/q\right)\left(\int_A^B \phi(t)\mathrm{d}t + P^{\delta_0}\right) \\ &= \frac{1}{q}\sum_{z=1}^{q} e\left(a_i a z^d/q\right)\upsilon_{f,i}(\boldsymbol{\beta}) + O\left(P^{\delta_0+\xi}\right),\end{aligned}$$

where in the last step we used the fact that $1 \leq q \leq P^{\xi}$. The proof is now complete. $\square$

By Lemma 3.7.2 and using a standard telescoping identity one has that

$$\begin{aligned}\mathcal{F}(\boldsymbol{\alpha}) - \mathcal{F}^{\star}(\boldsymbol{\beta}) &\ll P^{s-1}\left(|f_i - f_i^{\star}| + |g_j - g_j^{\star}| + |h_k - h_k^{\star}|\right) \\ &\ll P^{s-1+\delta_0+\xi}.\end{aligned}$$

Moreover one has

$$\text{meas}\left(\mathfrak{N}_{\xi}(q,a) \times \mathfrak{M}\right) \asymp P^{-d+\xi} \cdot P^{-\theta+\delta_0} = P^{-(\theta+d)+\delta_0+\xi}.$$

Next, note that one has $\mathcal{F}^{\star}(\boldsymbol{\beta}) = V(\boldsymbol{\beta})T(q,a)$. Integrating over the set $\mathfrak{N}_{\xi}(q,a) \times \mathfrak{M}$ against the measure $K_{\pm}(\alpha_{\theta})\mathrm{d}\boldsymbol{\alpha}$ and taking into account the preceding observations reveals

$$\int_{\mathfrak{M}}\int_{\mathfrak{N}_{\xi}(q,a)} \mathcal{F}(\boldsymbol{\alpha})K_{\pm}(\alpha_{\theta})\mathrm{d}\boldsymbol{\alpha} = T(q,a)\int_{\mathfrak{M}}\int_{\mathfrak{N}_{\xi}(q,a)} V(\boldsymbol{\beta})K_{\pm}(\alpha_{\theta})\mathrm{d}\boldsymbol{\beta} + E,$$

where

$$E = O\left(P^{s-(\theta+d)-1+2(\delta_0+\xi)}\right).$$

One can now sum over $1 \leq q \leq P^{\xi}$ and $1 \leq r \leq q$ to conclude that

$$\int_{\mathfrak{M}}\int_{\mathfrak{N}_{\xi}} \mathcal{F}(\boldsymbol{\alpha})K_{\pm}(\alpha_{\theta})\mathrm{d}\boldsymbol{\alpha} = \mathfrak{S}(P^{\xi})\mathfrak{J}^{\pm}(P^{\xi}, P^{\delta_0}) + O\left(P^{s-(\theta+d)-1+2\delta_0+4\xi}\right).$$

Recall from (3.2.7) that $\delta_0 = 2^{1-2\theta}$ and from (3.2.8) that $0 < \xi \leq \delta_0/8$. Recall that we assume $\theta > d+1 \geq 3$. Hence, for the error term in the above asymptotic formula one has

$$P^{s-(\theta+d)-1+2\delta_0+4\xi} \ll P^{s-(\theta+d)-1+\frac{5}{2}\delta_0} = o\left(P^{s-(\theta+d)}\right).$$

By Lemma 3.6.7 and Lemma 3.6.9 one has

$$\mathfrak{S}(P^{\xi})\mathfrak{J}^{\pm}(P^{\xi}, P^{\delta_0}) = 2\tau\mathfrak{J}_0\mathfrak{S}P^{s-(\theta+d)} + o\left(P^{s-(\theta+d)}\right).$$

Thus we conclude that

$$\int_{\mathfrak{P}} \mathcal{F}(\boldsymbol{\alpha})K_{\pm}(\alpha_{\theta})\mathrm{d}\boldsymbol{\alpha} = 2\tau\mathfrak{J}_0\mathfrak{S}P^{s-(\theta+d)} + o\left(P^{s-(\theta+d)}\right),$$

where recall that $\mathfrak{P} = \mathfrak{N}_{\xi} \times \mathfrak{M}$. Upon invoking (3.2.10) and taking into account Lemma 3.4.11 and Lemma 3.5.1, the proof of Theorem 3.1.2 is complete.

Appendix A

"Orthogonality " for inequalities

In this appendix we give a variant of Lemma 2.3.2 for a more general case that might be of some interest and usage in problems of counting solutions of inequalities in terms of mean values.

Let $R \in \mathbb{N}$. Suppose that

$$\boldsymbol{\phi} : \mathbb{R} \to \mathbb{R}^K \quad x \mapsto \boldsymbol{\phi}(x) = (\phi_1(x), \ldots, \phi_K(x))$$

is a $\mathbb{R}^R$-valued function of the real variable x. Furthermore, let $\delta_j > 0 \ (1 \leq j \leq R)$ be positive real numbers. We aim to count the number of integer solutions $\mathbf{x}$ of the simultaneous inequalities

$$\left| \sum_{i=1}^{s} (\phi_j(x_i) - \phi_j(x_{s+i})) \right| < \delta_j \qquad (1 \leq j \leq R). \tag{A.1}$$

Suppose that $I_1, I_2 \subset (0, \infty)$ are two bounded intervals. Fix a natural number $n \leq s$. Suppose that $\mathcal{S} \subset (0, \infty)^{2n}$ is a bounded set of lattice points. We write $V_s(I_1, I_2; \boldsymbol{\phi}, \boldsymbol{\delta})$ to denote the number of integer solutions of the simultaneous inequalities (A.1) with $x_i, x_{s+i} \in I_1 \ (1 \leq i \leq n)$ and $x_i, x_{s+i} \in I_2 \ (n+1 \leq i \leq s)$. Similarly, we write $V_s(\mathcal{S}, I_2; \boldsymbol{\phi}, \boldsymbol{\delta})$ to denote the number of integer solutions of the simultaneous inequalities (A.1) with $(x_1, \ldots, x_n, x_{s+1}, \ldots, x_{s+n}) \in \mathcal{S}$ and $x_i, x_{s+i} \in I_2 \ (n+1 \leq i \leq s)$. For ease of notation we set

$$\sigma_{s,j}(\mathbf{x}) = \sum_{i=1}^{s} (\phi_j(x_i) - \phi_j(x_{s+i})) \qquad (1 \leq j \leq R).$$

For any $\boldsymbol{\alpha} \in \mathbb{R}^R$ we define the exponential sums $H_i(\boldsymbol{\alpha}) = H(\boldsymbol{\alpha}; I_i; \boldsymbol{\phi})$ via

$$H(\boldsymbol{\alpha}; I_i; \boldsymbol{\phi}) = \sum_{x \in I_i} e(\alpha_1 \phi_1(x) + \cdots + \alpha_K \phi_K(x)) \qquad (i = 1, 2).$$

Moreover, for $\boldsymbol{\alpha} \in \mathbb{R}^R$ we put

$$H_{\mathcal{S}}(\boldsymbol{\alpha}) = \sum_{\mathbf{x} \in \mathcal{S}} e(\alpha_1 (\phi_1(x_1) - \phi_1(x_{s+1})) + \cdots + \alpha_R (\phi_R(x_n) - \phi_R(x_{s+n}))),$$

where the summation is over tuples $\mathbf{x} = (x_1, \ldots, x_n, x_{s+1}, \ldots, x_{s+n}) \in \mathcal{S}$. Finally, we define the numbers Δ_j via the relation $2\delta_j \Delta_j = 1 \ (1 \leq j \leq R)$. With this notation, we may now state the following lemma which is a variant of Lemma 2.3.2 in a more general form.

Lemma A.1. *One has*

(i)
$$V_s\left(\mathcal{S}, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right) \ll \delta_1 \cdots \delta_R \int_{-\Delta_R}^{\Delta_R} \cdots \int_{-\Delta_1}^{\Delta_1} \left|H_{\mathcal{S}}(\boldsymbol{\alpha}) H_2(\boldsymbol{\alpha})^{2s-2n}\right| \mathrm{d}\boldsymbol{\alpha};$$

(ii)
$$V_s\left(I_1, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right) \asymp \delta_1 \cdots \delta_R \int_{-\Delta_R}^{\Delta_R} \cdots \int_{-\Delta_1}^{\Delta_1} \left|H_1(\boldsymbol{\alpha})^{2n} H_2(\boldsymbol{\alpha})^{2s-2n}\right| \mathrm{d}\boldsymbol{\alpha}.$$

The implicit constants in the above estimates are independent of $I_1, I_2, \mathcal{S}, \boldsymbol{\delta}$ *and* $\boldsymbol{\phi}$.

The proof makes use of the following pair of functions. For $x \in \mathbb{R}$ we define the functions

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & \text{when } x \neq 0, \\ 1, & \text{when } x = 0, \end{cases}$$

and

$$\Lambda(x) = \max\{0, 1 - |x|\}.$$

Moreover, we set $K(x) = \operatorname{sinc}^2(x)$. It is well known, see for example [29, Lemma 20.1] that these two functions are the Fourier transform of each other. Namely, for $x, \xi \in \mathbb{R}$ one has

$$K(\xi) = \int_{-\infty}^{\infty} e(-x\xi)\Lambda(x)\mathrm{d}x \quad \text{and} \quad \Lambda(x) = \int_{-\infty}^{\infty} e(x\xi)K(\xi)\mathrm{d}\xi. \tag{A.2}$$

Moreover, we make use of Jordan's inequality which states that for $0 < x \leq \frac{\pi}{2}$ one has

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1, \tag{A.3}$$

where the equality holds if and only if $x = \frac{\pi}{2}$. A proof of this inequality can be bound in [48, p. 33].

Proof of Lemma A.1. (i). By (A.3) one has for $|x| < \frac{1}{2}$ that

$$K(x) > \left(\frac{2}{\pi}\right)^2.$$

For ease of notation we put

$$\xi_j = \frac{1}{2\delta_j}\sigma_{s,j}(\mathbf{x}) \qquad (1 \leq j \leq R).$$

Let $\mathbf{x}$ be a solution of the simultaneous inequalities (A.1) counted by $V_s\left(\mathcal{S}, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right)$. Then for each index $1 \leq j \leq R$ one has

$$\frac{\pi^2}{4} K\left(\xi_j\right) > 1.$$

Hence, by summing over tuples $\mathbf{x}$ with $(x_i, x_{s+i}) \in \mathcal{S}$ $(1 \leq i \leq n)$ and $x_i, x_{s+i} \in I_2$ $(n + 1 \leq$

$i \leq s$) one has that

$$\begin{aligned} V_s\left(\mathcal{S}, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right) &\leq \sum_{\mathbf{x}} \prod_{j=1}^{R} \frac{\pi^2}{4} K\left(\xi_j\right) \\ &= \left(\frac{\pi^2}{4}\right)^R \sum_{\mathbf{x}} \prod_{j=1}^{R} K\left(\xi_j\right). \end{aligned} \tag{A.4}$$

Then, by invoking (A.2) we infer that for each $1 \leq j \leq R$ one has

$$K(\xi_j) = \int_{-\infty}^{\infty} e(-u\xi_j)\Lambda(u)\mathrm{d}u = \int_{-\infty}^{\infty} e(u\xi_j)\Lambda(-u)\mathrm{d}u.$$

So, by making the change of variables $u_j = 2\delta_j\alpha_j \ (1 \leq j \leq R)$ we deduce that

$$\prod_{j=1}^{R} K(\xi_j) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e(u_1\xi_1)\Lambda(-u_1)\cdots e(u_R\xi_R)\Lambda(-u_R)\mathrm{d}\mathbf{u}$$

$$= 2^R\delta_1\cdots\delta_R \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e\left(\alpha_1\sigma_{s,1}(\mathbf{x}) + \cdots + \alpha_R\sigma_{s,R}(\mathbf{x})\right) \Lambda\left(-2\delta_1\alpha_1\right)\cdots\Lambda\left(-2\delta_R\alpha_R\right) \mathrm{d}\boldsymbol{\alpha}.$$

For each index $1 \leq i \leq 2s$ we write $\boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_i)$ to denote the standard dot product in the space $\mathbb{R}^R$, namely

$$\boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_i) = \alpha_1\phi_1(x_i) + \cdots + \alpha_K\phi_k(x_i).$$

With this notation one has

$$e\left(\alpha_1\sigma_{s,1}(\mathbf{x}) + \cdots + \alpha_R\sigma_{s,R}(\mathbf{x})\right) = e\left(\boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_1) + \cdots - \boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_{2s})\right).$$

So we infer that $\prod_{j=1}^{R} K(\xi_j)$ is equal to

$$2^R\delta_1\cdots\delta_R \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e\left(\boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_1) + \cdots - \boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_{2s})\right) \Lambda\left(-2\delta_1\alpha_1\right)\cdots\Lambda\left(-2\delta_R\alpha_R\right) \mathrm{d}\boldsymbol{\alpha}.$$

One can now sum over tuples $\mathbf{x} = (x_1, \ldots, x_{2s})$ with $(x_1, \ldots, x_n, x_{s+1}, \ldots, x_{s+n}) \in \mathcal{S}$ and $x_i, x_{s+i} \in I_2 \ (n+1 \leq i \leq s)$. Since the integrals are absolutely convergent one can interchange the order of integration with the finite sums. Note here that

$$\sum_{\mathbf{x}} e\left(\boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_1) + \cdots - \boldsymbol{\alpha}\cdot\boldsymbol{\phi}(x_{2s})\right) = H_{\mathcal{S}}(\boldsymbol{\alpha})H_2(\boldsymbol{\alpha})^{2s-2n}.$$

Furthermore, note that for $|\alpha_j| > \Delta_j = 2\delta_j^{-1}$ one has $\Lambda(-2\delta_j\alpha_j) = 0$ and for all $\boldsymbol{\alpha}$ one has

$$\left|H_{\mathcal{S}}(\boldsymbol{\alpha})H_2(\boldsymbol{\alpha})^{2s-2n}\right| \geq 0.$$

Keep in mind that for any x one has $0 \leq \Lambda(x) \leq 1$. Hence by (A.4) and the triangle inequality we deduce that

$$\begin{aligned}
V_s\left(\mathcal{S}, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right) &= \left(\frac{\pi^2}{2}\right)^R \delta_1 \cdots \delta_R \times \\
&\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_{\mathcal{S}}(\boldsymbol{\alpha}) H_2(\boldsymbol{\alpha})^{2s-2n} \Lambda\left(-2\delta_1\alpha_1\right) \cdots \Lambda\left(-2\delta_R\alpha_R\right) \mathrm{d}\boldsymbol{\alpha} \\
&\leq \left(\frac{\pi^2}{2}\right)^R \delta_1 \cdots \delta_R \times \\
&\times \int_{-\Delta_R}^{\Delta_R} \cdots \int_{-\Delta_1}^{\Delta_1} \left|H_{\mathcal{S}}(\boldsymbol{\alpha}) H_2(\boldsymbol{\alpha})^{2s-2n}\right| \Lambda\left(-2\delta_1\alpha_1\right) \cdots \Lambda\left(-2\delta_R\alpha_R\right) \mathrm{d}\boldsymbol{\alpha} \\
&\ll \delta_1 \cdots \delta_R \int_{-\Delta_R}^{\Delta_R} \cdots \int_{-\Delta_1}^{\Delta_1} \left|H_{\mathcal{S}}(\boldsymbol{\alpha}) H_2(\boldsymbol{\alpha})^{2s-2n}\right| \mathrm{d}\boldsymbol{\alpha},
\end{aligned}$$

which establishes the upper bound.

We now prove (ii). For the upper bound we argue in a similar fashion as in (i), whereas now we use the product $H_1(\boldsymbol{\alpha})^{2n} H_2(\boldsymbol{\alpha})^{2s-2n}$. So we focus on proving the lower bound. Let **x** be a tuple counted by $V_s\left(I_1, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right)$. Then by the definition of the function $x \mapsto \Lambda(x)$, one has for each index $1 \leq j \leq R$ that

$$0 < \Lambda\left(2\xi_j\right) < 1.$$

One can now sum over tuples **x** with $x_i, x_{s+i} \in I_1 \ \ (1 \leq i \leq n)$ and $x_i, x_{s+i} \in I_2 \ \ (n+1 \leq i \leq s)$ to obtain

$$V_s\left(I_1, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right) \geq \sum_{\mathbf{x}} \prod_{j=1}^{R} \Lambda(2\xi_j). \tag{A.5}$$

Invoking (A.2) we infer for each index $1 \leq j \leq R$ that

$$\Lambda(2\xi_j) = \int_{-\infty}^{\infty} e(2u\xi_j) K(u) \mathrm{d}u.$$

As before, one may now sum over the $x_i \in I_i$ and interchange the order of summation and integration to get by (A.5) that $V_s\left(I_1, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right)$ equals to

$$\delta_1 \cdots \delta_R \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{\mathbf{x}} e\left(\boldsymbol{\alpha} \cdot \boldsymbol{\phi}(x_1) + \cdots - \boldsymbol{\alpha} \cdot \boldsymbol{\phi}(x_{2s})\right) K(\delta_1\alpha_1) \cdots K(\delta_R\alpha_R) \mathrm{d}\boldsymbol{\alpha}.$$

Since we assume that x_i and x_{s+i} belong to the same interval I_1 for each $1 \leq i \leq n$, one has that

$$\sum_{\mathbf{x}} e\left(\boldsymbol{\alpha} \cdot \boldsymbol{\phi}(x_1) + \cdots - \boldsymbol{\alpha} \cdot \boldsymbol{\phi}(x_{2s})\right) = \left|H_1(\boldsymbol{\alpha})^{2n} H_2(\boldsymbol{\alpha})^{2s-2n}\right|.$$

By (A.3) one has for $|\alpha_j| < (2\delta_j)^{-1} = \Delta_j$ that $K(\delta_j\alpha_j) < 4/\pi^2$. Therefore, we deduce that

$$V_s\left(I_1, I_2; \boldsymbol{\phi}, \boldsymbol{\delta}\right) \gg \delta_1 \cdots \delta_R \int_{-\Delta_R}^{\Delta_R} \cdots \int_{-\Delta_1}^{\Delta_1} \left|H_1(\boldsymbol{\alpha})^{2n} H_2(\boldsymbol{\alpha})^{2s-2n}\right|.$$

The proof is now complete. $\square$

We give a variant of Lemma A.1. Let $\mathcal{I} = \{I_1, \cdots, I_{2s}\}$ be a collection of bounded intervals $I_i \subset (0, \infty)$ for all i. We aim to count the number of integer solutions **x** with $x_i \in I_i$ $(1 \leq i \leq 2s)$, of the simultaneous inequalities (A.1). We denote the number of such solutions by $V_s(\mathcal{I}; \boldsymbol{\phi}, \boldsymbol{\delta})$. For any $\boldsymbol{\alpha} \in \mathbb{R}^K$ we define the exponential sums $H_i(\boldsymbol{\alpha}) = H(\boldsymbol{\alpha}; I_i; \boldsymbol{\phi})$ via

$$H(\boldsymbol{\alpha}; I_i; \boldsymbol{\phi}) = \sum_{x \in I_i} e(\alpha_1 \phi_1(x) + \cdots + \alpha_K \phi_K(x)) \qquad (1 \leq i \leq 2s).$$

Moreover, we define the mean value

$$\mathfrak{J}_s(\mathcal{I}; \boldsymbol{\phi}, \boldsymbol{\Delta}) = \int_{-\Delta_K}^{\Delta_K} \cdots \int_{-\Delta_1}^{\Delta_1} |H_1(\boldsymbol{\alpha}) \cdots H_{2s}(\boldsymbol{\alpha})| \, \mathrm{d}\boldsymbol{\alpha},$$

where the numbers Δ_j are defined via the relation $2\delta_j \Delta_j = 1$ $(1 \leq j \leq K)$. With this notation, we may now state the following lemma which is a variant of Lemma A.1 and one may prove arguing in a similar fashion.

Lemma A.2. *One has*

$$V_s(\mathcal{I}; \boldsymbol{\phi}, \boldsymbol{\delta}) \ll \delta_1 \cdots \delta_K \mathfrak{J}_s(\mathcal{I}; \boldsymbol{\phi}, \boldsymbol{\Delta}).$$

Moreover, if $I_i = I_{s+i}$ for $1 \leq i \leq s$, then one has

$$\delta_1 \cdots \delta_K \mathfrak{J}_s(\mathcal{I}; \boldsymbol{\phi}, \boldsymbol{\Delta}) \ll V_s(\mathcal{I}; \boldsymbol{\phi}, \boldsymbol{\delta}).$$

The implicit constants in the above estimates are independent of $\mathcal{I}$, $\boldsymbol{\phi}$, and $\boldsymbol{\delta}$.

Appendix B

The mean value estimate for the complete exponential sum

This appendix contains the details required if one wishes to obtain a mean value estimate as in Theorem 2.1.4 for the exponential sum

$$f(\boldsymbol{\alpha}; P) = \sum_{P < x \leq 2P} e\left(\alpha_1 x + \cdots + \alpha_n x^n + \alpha_\theta x^\theta\right),$$

where $n = \lfloor \theta \rfloor$. For technical reasons we assume that the exponential sum f is defined over a dyadic interval $(P, 2P]$. If the exponential sum f is defined over an interval of the shape $[1, P]$, then by making abuse of notation one may split this into $O(\log P)$ dyadic intervals of the shape $(P, 2P]$. The $\log P$-loss is absorbed into the ϵ-loss P^ϵ. We demonstrate the following.

Theorem B.1. *Suppose that $\theta > 2$ is real and non-integral and write $n = \lfloor \theta \rfloor$. Let $\kappa \geq 1$ be a real number. Suppose further that $s \geq \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ is a natural number. Then for any fixed $\epsilon > 0$ one has*

$$\int_{-\kappa}^{\kappa} \int_{[0,1)^n} |f(\boldsymbol{\alpha}; P)|^{2s} \, \mathrm{d}\boldsymbol{\alpha} \ll \kappa P^{2s - \frac{1}{2}n(n+1) - \theta + \epsilon}.$$

The implicit constant in the above estimate may depend on ϵ, θ, and s, but not on κ and P. Furthermore, for $s > \frac{1}{2}(\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2)$ one can take $\epsilon = 0$.

First we need a replacement of Lemma 2.3.2. From now on we set $k = \lfloor 2\theta \rfloor + 1$. For each index j with $n + 1 \leq j \leq k$ we define the binomial coefficients

$$b_j = \binom{\theta}{j} = \frac{\theta(\theta - 1) \cdots (\theta - j + 1)}{j\,!}.$$

For fixed θ one may treat the b_j as being of size $O(1)$. For a tuple $\mathbf{h} = (h_{n+1}, \ldots, h_k)$ we write $\mathcal{H}(\mathbf{h}) = \mathcal{H}(h_{n+1}, \ldots, h_k)$ to denote the expression

$$\mathcal{H}(h_{n+1}, \ldots, h_k) = b_{n+1} P^{\theta - (n+1)} h_{n+1} + \cdots + b_k P^{\theta - k} h_k. \tag{B.1}$$

Lemma B.2. *Let L be a given positive real number, and t be a given natural number. We write*

$T(P)$ to denote the number of integer solutions of the inequality

$$|\mathcal{H}(\boldsymbol{h})| \leq L$$

in the variables h_j satisfying $|h_j| \leq tP^{j/2} \quad (n+1 \leq j \leq k)$. Then one has

$$T(P) \ll P^{\frac{k(k+1)}{4} - \frac{n(n+1)}{4} - \theta + \frac{1}{2}},$$

where the implicit constant depends on L, t and θ.

Proof. The proof is similar to [1, Lemma 1]. One may rewrite the inequality under investigation, in the shape

$$b_{n+1}P^{\theta-(n+1)}h_{n+1} + \cdots + b_k P^{\theta-k}h_k = L\gamma, \tag{B.2}$$

for some γ satisfying $|\gamma| \leq 1$. From this equation we get that

$$h_{n+1} = -b_{n+1}^{-1}P^{n+1-\theta}\sum_{\ell=n+2}^{k} b_\ell P^{\theta-\ell}h_\ell + L\gamma b_{n+1}^{-1}P^{n+1-\theta}. \tag{B.3}$$

By our hypothesis one has $|h_\ell| \ll P^{\frac{\ell}{2}}$ for all $n+1 \leq \ell \leq k$. Hence, the first term appearing on the right hand side of (B.3) is bounded above by

$$\ll P^{n+1-\theta}\sum_{\ell=n+2}^{k}|h_\ell|P^{\theta-\ell} \ll P^{n+1}\sum_{\ell=n+2}^{k}\frac{1}{P^{\frac{\ell}{2}}} \ll P^{n+1-\frac{n+2}{2}} \ll P^{\frac{n}{2}}.$$

Recall now that $n = \lfloor \theta \rfloor \geq 2$. So one has $n+1-\theta < \frac{n}{2}$. Hence, the second term on the right hand side of (B.3) is $\ll P^{\frac{n}{2}}$. Thus, we deduce

$$|h_{n+1}| \ll P^{\frac{n}{2}}.$$

So, the unknown h_{n+1} assumes at most $O\left(P^{\frac{n}{2}}\right)$ possible values.

Fix now an index j_0 with $n+2 \leq j_0 \leq k$. Then, the equation (B.2) can be equivalently written as

$$h_{j_0} + b_{j_0}^{-1}P^{j_0-\theta}\sum_{\ell=n+1}^{j_0-1} b_\ell P^{\theta-\ell}h_\ell = -b_{j_0}^{-1}P^{j_0-\theta}\sum_{\ell=j_0+1}^{k} b_\ell P^{\theta-\ell}h_\ell + L\gamma b_{j_0}^{-1}P^{j_0-\theta}. \tag{B.4}$$

We put

$$A_{j_0} = b_{j_0}^{-1}P^{j_0-\theta}\sum_{\ell=n+1}^{j_0-1} b_\ell P^{\theta-\ell}h_\ell,$$

and

$$B_{j_0} = -b_{j_0}^{-1}P^{j_0-\theta}\sum_{\ell=j_0+1}^{k} b_\ell P^{\theta-\ell}h_\ell + L\gamma b_{j_0}^{-1}P^{j_0-\theta}.$$

Then, by (B.4) we obtain that

$$|h_{j_0} + A_{j_0}| \leq |B_{j_0}|. \tag{B.5}$$

By the definition of the expression B_{j_0} we have that

$$|B_{j_0}| \leq |b_{j_0}^{-1}| \left(P^{j_0-\theta} \sum_{\ell=j_0+1}^{k} |b_\ell|\, P^{\theta-\ell} |h_\ell| + L|\gamma| P^{j_0-\theta} \right).$$

We now bound the expression lying in the parentheses on right hand side of the above inequality. Using again our assumption that $|h_\ell| \ll P^{\ell/2}$ one has

$$\begin{aligned} P^{j_0-\theta} \sum_{\ell=j_0+1}^{m} |b_\ell|\, P^{\theta-\ell} |h_\ell| + L|\gamma| P^{j_0-\theta} &= P^{j_0} \sum_{\ell=j_0+1}^{k} |b_\ell| P^{-\ell} |h_\ell| + L|\gamma| P^{j_0-\theta} \\ &\ll P^{j_0} \sum_{\ell=j_0+1}^{k} \frac{1}{P^{\ell/2}} + P^{j_0-\theta} \\ &\ll P^{\frac{j_0-1}{2}} + P^{j_0-\theta}. \end{aligned}$$

So one has

$$|B_{j_0}| \ll P^{\frac{j_0-1}{2}} + P^{j_0-\theta}.$$

Thus by (B.5) we infer that

$$|h_{j_0} + A_{j_0}| \ll P^{\frac{j_0-1}{2}} + P^{j_0-\theta}.$$

Hence, by the triangle inequality we deduce that for each fixed index j_0 with $n+2 \leq j_0 \leq k$ one has

$$|h_{j_0}| \leq |h_{j_0} + A_{j_0}| + |A_{j_0}| \ll P^{\frac{j_0-1}{2}} + P^{j_0-\theta} + |A_{j_0}|. \tag{B.6}$$

Let us now fix a value for h_{n+1}. As we proved in the beginning, the variable h_{n+1} can assume at most $O\left(P^{\frac{n}{2}}\right)$ values. For the variables h_j $(n+2 \leq j \leq k-2)$ one can argue inductively as follows. Let j_0 be such an index and suppose that the variables h_j with $n+1 \leq j < j_0$ have been fixed. In such a case, by the definition of A_{j_0} one has that

$$|A_{j_0}| \ll P^{j_0} \sum_{\ell=n+1}^{j_0-1} \frac{1}{P^\ell} \ll P^{j_0-(n+1)} \ll P^{\frac{j_0-1}{2}}.$$

Appealing to (B.6) we get that

$$|h_{j_0}| \ll P^{\frac{j_0-1}{2}} + P^{j_0-\theta} \ll P^{\frac{j_0-1}{2}},$$

where in the second inequality we used the fact that $j_0 \leq k-2 = \lfloor 2\theta \rfloor$. Therefore, we deduce that the unknowns h_j $(n+2 \leq j \leq k-2)$ assume at most $O\left(P^{\frac{j-1}{2}}\right)$ values.

We are now left to deal with the last two variables, namely with h_{k-1} and h_k. Assuming that the variables h_j $(n+1 \leq j \leq k-2)$ have been fixed and recalling that $n = \lfloor \theta \rfloor$ one has by the definition of A_{j_0} with $j_0 = k-1$ that

$$|A_{k-1}| \ll P^{k-1} \sum_{\ell=n+1}^{k-2} \frac{1}{P^\ell} \ll P^{(k-1)-(n+1)} \ll P^{(k-1)-\theta}.$$

Note here that

$$k-1-\theta=\lfloor 2\theta\rfloor-\theta>\frac{\lfloor 2\theta\rfloor-1}{2}=\frac{k-2}{2}.$$

Hence, appealing to (B.6) we now infer that

$$|h_{k-1}|\ll P^{\frac{(k-1)-1}{2}}+P^{(k-1)-\theta}+P^{(k-1)-\theta}\ll P^{(k-1)-\theta}.$$

Thus, the unknown h_{k-1} assumes at most $O\left(P^{(k-1)-\theta}\right)$ possible values. Finally, again by our initial assumption, we know that the variable h_k assumes at most $O\left(P^{\frac{k}{2}}\right)$ values.

Summarising the above we have showed the following.

- The unknown variable h_{n+1} can assume $O\left(P^{\frac{n}{2}}\right)$ values.
- The unknown variables $h_j\ (n+2\leq j\leq k-2)$ can assume $O\left(P^{\frac{j-1}{2}}\right)$ values.
- The unknown variable h_{k-1} can assume $O\left(P^{(k-1)-\theta}\right)$ values.
- The unknown variable h_k can assume $O\left(P^{\frac{k}{2}}\right)$ values.

Collecting together the above conclusions, we can now deduce that

$$\begin{aligned}T(P)&\ll P^{\frac{n}{2}}\cdot P^{\frac{n+1}{2}}\cdots P^{\frac{k-3}{2}}\cdot P^{(k-1)-\theta}\cdot P^{\frac{k}{2}}\\&\ll P^{\frac{1}{4}k(k+1)-\frac{1}{4}n(n+1)-\theta+\frac{n+1}{2}},\end{aligned}$$

since one can easily verify that

$$\begin{aligned}\frac{n}{2}+\frac{n+1}{2}+\cdots+\frac{k-3}{2}+(k-1)-\theta+\frac{k}{2}&=\frac{1}{4}k(k+1)-\frac{1}{4}n(n-1)-\theta+\frac{1}{2}\\&=\frac{1}{4}k(k+1)-\frac{1}{4}n(n+1)-\theta+\frac{n+1}{2}.\end{aligned}$$

The proof of the lemma is now complete. □

Next, we fix some notation for the rest of this section. From now on when applying Lemma A.1 we take

$$R=n+1,\qquad \boldsymbol{\phi}=\left(x,\ldots,x^n,x^\theta\right).$$

Furthermore, recall that $k=\lfloor 2\theta\rfloor+1$ and $n=\lfloor\theta\rfloor$. For each $j\in\{1,\ldots,n,\theta\}$ we set

$$\sigma_{s,j}(\mathbf{x})=\sum_{i=1}^{s}\left(x_i^j-x_{s+i}^j\right) \tag{B.7}$$

Moreover, we write $f(\boldsymbol{\alpha};P)=f(\boldsymbol{\alpha})$. We now embark to the proof of Theorem B.1.

Proof of Theorem B.1. We split the proof into several steps for a better presentation. From now on we set $I=(P,2P]$.

Step 1: The underlying Diophantine system. Set $\boldsymbol{\delta}_1=\left(\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2\kappa}\right)$. We apply Lemma

A.1 with $I_1 = I_2 = I$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_1$. So, one has

$$\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\int_{[-1,1]^n} |f(\boldsymbol{\alpha})|^{2s}\,\mathrm{d}\hat{\boldsymbol{\alpha}}\mathrm{d}\alpha_\theta \ll V_s\left(I;\boldsymbol{\delta}_1\right), \tag{B.8}$$

where $\mathrm{d}\hat{\boldsymbol{\alpha}}\mathrm{d}\alpha_\theta$ stands for $\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_n\mathrm{d}\alpha_\theta$ and $V_s(I;\boldsymbol{\delta}_1)$ denotes the number of integer solutions of the system

$$\begin{cases} |\sigma_{s,j}(\mathbf{x})| < \frac{1}{2} & (1 \leq j \leq n) \\ |\sigma_{s,\theta}(\mathbf{x})| < \frac{1}{2\kappa}, \end{cases}$$

with $P < \mathbf{x} \leq 2P$, where recall from (B.7) the definition of $\sigma_{s,j}(\mathbf{x})$.

Set $\boldsymbol{\delta}_2 = \left(\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2}\right)$. Since $\kappa \geq 1$ we plainly have that

$$V_s\left(I;\boldsymbol{\delta}_1\right) \leq V_s\left(I;\boldsymbol{\delta}_2\right),$$

where $V_s(I;\boldsymbol{\delta}_2)$ denotes the number of integer solutions of the system

$$\begin{cases} |\sigma_{s,j}(\mathbf{x})| < \frac{1}{2} & (1 \leq j \leq n) \\ |\sigma_{s,\theta}(\mathbf{x})| < \frac{1}{2}, \end{cases}$$

with $P < \mathbf{x} \leq 2P$. By the estimate (B.8) we infer that

$$\int_{-\kappa}^{\kappa}\int_{[-1,1]^n} |f(\boldsymbol{\alpha})|^{2s}\,\mathrm{d}\hat{\boldsymbol{\alpha}}\mathrm{d}\alpha_\theta \ll \kappa V_s\left(I;\boldsymbol{\delta}_2\right). \tag{B.9}$$

In view of (B.9) we are now aiming to bound from above the counting function $V_s\left(\mathcal{I};\boldsymbol{\delta}_2\right)$.

From now on, the mean values we encounter are independent of κ. For this reason, and for ease of the notation, we use the symbol $\oint$ to denote integration over the $(n+1)$-cube $[-1,1]^{n+1}$. We define the interval

$$\widetilde{I} = \left(P,\ P + \left(\lfloor\sqrt{P}\rfloor + 1\right)\sqrt{P}\right],$$

and note that $I \subset \widetilde{I}$. Moreover, we define the exponential sum

$$\widetilde{f}(\boldsymbol{\alpha}) = \sum_{x\in\widetilde{I}} e\left(\alpha_1 x + \cdots + \alpha_n x^n + \alpha_\theta x^\theta\right).$$

Since $I \subset \widetilde{I}$ one has that

$$V_s\left(\mathcal{I};\boldsymbol{\delta}_2\right) \leq V_s\left(I^{(n+1)},\widetilde{I};\boldsymbol{\delta}_2\right),$$

where $V_s\left(I^{(n+1)},\widetilde{I};\boldsymbol{\delta}_2\right)$ denotes the number of integer solutions of the system

$$\begin{cases} \left|\sum_{i=1}^{n+1}\left(x_i^\theta - x_{s+i}^\theta\right) + \sum_{i=n+2}^{s}\left(x_i^\theta - x_{s+i}^\theta\right)\right| < \dfrac{1}{2} \\ \left|\sum_{i=1}^{n+1}\left(x_i^j - x_{s+i}^j\right) + \sum_{i=n+2}^{s}\left(x_i^j - x_{s+i}^j\right)\right| < \dfrac{1}{2} & (1 \leq j \leq n), \end{cases}$$

with $x_i, x_{s+i} \in I$ $(1 \le i \le n+1)$ and $x_i, x_{s+i} \in \widetilde{I}$ $(n+2 \le i \le t)$. Applying again Lemma A.1 with $\boldsymbol{\delta} = \boldsymbol{\delta}_2$ we obtain that

$$V_s\left(I^{(n+1)}, \widetilde{I}; \boldsymbol{\delta}_2\right) \ll \oint |f(\boldsymbol{\alpha})|^{2(n+1)} |\widetilde{f}(\boldsymbol{\alpha})|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha}.$$

Combining the above we infer that

$$V_s(I; \boldsymbol{\delta}_2) \ll \oint |f(\boldsymbol{\alpha})|^{2(n+1)} |\widetilde{f}(\boldsymbol{\alpha})|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha}. \tag{B.10}$$

Step 2: Breaking the interval $\widetilde{I}$ into short intervals. For a natural number $\ell \ge 1$ we write

$$P_\ell = P + (\ell - 1)\sqrt{P}, \tag{B.11}$$

and set $\widetilde{I}_\ell = (P_\ell, P_{\ell+1}]$. Note that $\widetilde{I}_\ell$ forms a cover of the interval $\widetilde{I}$, consisting of subintervals of length $\sqrt{P}$. We record this in the following inclusion

$$I \subset \widetilde{I} \subset \bigcup_{\ell=1}^{\lfloor \sqrt{P} \rfloor + 1} \widetilde{I}_\ell. \tag{B.12}$$

We write $\widetilde{f}_\ell(\boldsymbol{\alpha})$ to denote the exponential sum given by

$$\widetilde{f}_\ell(\boldsymbol{\alpha}) = \sum_{x \in \widetilde{I}_\ell} e(\alpha_1 x + \cdots + \alpha_n x^n + \alpha_\theta x^\theta). \tag{B.13}$$

Incorporating the exponential sum $\widetilde{f}_\ell(\boldsymbol{\alpha})$ we deduce by the triangle inequality followed by an application of Hölder's inequality that

$$\begin{aligned}
&\oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left|\widetilde{f}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha} \\
&\le \oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left(\sum_{\ell=1}^{\lfloor \sqrt{P} \rfloor + 1} \left|\widetilde{f}_\ell(\boldsymbol{\alpha})\right|\right)^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha} \\
&\le \left(\lfloor \sqrt{P} \rfloor + 1\right)^{2s-2(n+1)-1} \sum_{\ell=1}^{\lfloor \sqrt{P} \rfloor + 1} \oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left|\widetilde{f}_\ell(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha}.
\end{aligned}$$

Thus, for some ℓ_0 with $1 \le \ell_0 \le \lfloor \sqrt{P} \rfloor + 1$ one has

$$\begin{aligned}
&\oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left|\widetilde{f}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha} \\
&\ll \left(\lfloor \sqrt{P} \rfloor + 1\right)^{2s-2(n+1)} \oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left|\widetilde{f}_{\ell_0}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha}.
\end{aligned}$$

Combining now the above with (B.10) we deduce that

$$\begin{aligned} V_s\left(I;\boldsymbol{\delta}_2\right) &\ll \left(\lfloor\sqrt{P}\rfloor+1\right)^{2s-2(n+1)} \oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left|\widetilde{f}_{\ell_0}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha} \\ &\ll P^{s-(n+1)} \oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left|\widetilde{f}_{\ell_0}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha}. \end{aligned} \tag{B.14}$$

Now we apply Lemma A.1 with $I_1 = I,\ I_2 = \widetilde{I}_{\ell_0}$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_2$. So one has

$$\oint |f(\boldsymbol{\alpha})|^{2(n+1)} \left|\widetilde{f}_{\ell_0}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha} \ll V_s\left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2\right), \tag{B.15}$$

where we write $V_s\left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2\right)$ to denote the number of integer solutions of the system

$$\begin{cases} \left|\sum_{i=1}^{n+1}\left(x_i^\theta - x_{s+i}^\theta\right) + \sum_{i=n+2}^{s}\left(x_i^\theta - x_{s+i}^\theta\right)\right| < \dfrac{1}{2} \\ \left|\sum_{i=1}^{n+1}\left(x_i^j - x_{s+i}^j\right) + \sum_{i=n+2}^{s}\left(x_i^j - x_{s+i}^j\right)\right| < \dfrac{1}{2} \qquad (1 \le j \le n), \end{cases} \tag{B.16}$$

with $x_i, x_{s+i} \in I\ (1 \le i \le n+1)$ and $x_i, x_{s+i} \in \widetilde{I}_{\ell_0}\ (n+2 \le i \le s)$. Putting together (B.14) and (B.15) we deduce that

$$V_s\left(I;\boldsymbol{\delta}_2\right) \ll P^{s-(n+1)} V_s\left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2\right). \tag{B.17}$$

Step 3: A diminishing ranges type argument. Recall that $\widetilde{I}_{\ell_0} = (P_{\ell_0}, 2P_{\ell_0}]$, where we write $P_{\ell_0} = P + (\ell_0 - 1)\sqrt{P}$. We now set

$$y_i = x_i - P_{\ell_0} \qquad (n+2 \le i \le s).$$

Clearly one has $0 < y_i \le \sqrt{P}$. Invoking the Binomial theorem, we see that a $2s$ tuple **x** satisfies system the (B.16), if and only if it satisfies the system

$$\begin{cases} \left|\sum_{i=1}^{n+1}\left(x_i^\theta - x_{s+i}^\theta\right) + \sum_{i=n+2}^{s}\left(\left(y_i + P_{\ell_0}\right)^\theta - \left(y_{s+i} + P_{\ell_0}\right)^\theta\right)\right| < \dfrac{1}{2} \\ \left|\sum_{i=1}^{n+1}\left(x_i^j - x_{s+i}^j\right) + \sum_{i=n+2}^{s}\left(y_i^j - y_{s+i}^j\right)\right| < \dfrac{1}{2} \qquad (1 \le j \le n). \end{cases} \tag{B.18}$$

For the sake of clarity, we note here that in the above system one has

$$P < x_i, x_{s+i} \le 2P \quad (1 \le i \le n+1) \quad \text{and} \quad 0 < y_i, y_{s+i} \le \sqrt{P} \quad (n+2 \le i \le s).$$

We now focus on the inequality of degree θ of the system (B.18). Since $P \gg P_{\ell_0} \gg \sqrt{P}$, an application of the mean value theorem of differential calculus yields for each index $n+2 \le i \le s$ that

$$\left|\left(y_i + P_{\ell_0}\right)^\theta - \left(y_{s+i} + P_{\ell_0}\right)^\theta\right| \asymp P_{\ell_0}^{\theta-1}|y_i - y_{s+1}| \ll P^{\theta-1/2}.$$

Hence we obtain that

$$\left| \sum_{i=n+2}^{s} \left((y_i + P_{\ell_0})^\theta - (y_{s+i} + P_{\ell_0})^\theta \right) \right| \ll P^{\theta - 1/2}.$$

Returning now to the system (B.18) one has

$$\left| \sum_{i=1}^{n+1} \left(x_i^\theta - x_{s+i}^\theta \right) \right| \ll P^{\theta-1/2}, \tag{B.19}$$

where $x_i, x_{s+i} \in I$. For $x \in I = (P, 2P]$ and using the mean value theorem of the differential calculus one has

$$|x_i^\theta - x_{s+i}^\theta| \asymp P^{\theta-1} |x_i - x_{s+i}| \qquad (1 \leq i \leq n+1).$$

Hence, by (B.19) we infer that

$$\left| \sum_{i=1}^{n+1} (x_i - x_{s+i}) \right| \ll P^{1/2}.$$

In other words, there exists a positive real number $C > 0$ depending at most on s and θ, such that

$$\left| \sum_{i=1}^{n+1} (x_i - x_{s+i}) \right| \leq C P^{1/2}.$$

Summarizing, we have showed that

$$V_s \left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2 \right) \ll V_s^\star \left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2 \right), \tag{B.20}$$

where $V_s^\star \left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2 \right)$ denotes the number of integer solutions of the system

$$\begin{cases} \left| \sum_{i=1}^{n+1} \left(x_i^\theta - x_{s+i}^\theta \right) + \sum_{i=n+2}^{s} \left(x_i^\theta - x_{s+i}^\theta \right) \right| < \dfrac{1}{2} \\ \left| \sum_{i=1}^{n+1} \left(x_i^j - x_{s+i}^j \right) + \sum_{i=n+2}^{s} \left(x_i^j - x_{s+i}^j \right) \right| < \dfrac{1}{2} \qquad (1 \leq j \leq n), \end{cases} \tag{B.21}$$

with

$$\begin{cases} x_i, x_{s+i} \in I \qquad (1 \leq i \leq n+1), \\ \left| \sum_{i=1}^{n+1} (x_i - x_{s+i}) \right| \leq C P^{1/2}, \\ x_i, x_{s+i} \in \widetilde{I}_{\ell_0} \qquad (n+2 \leq i \leq s). \end{cases} \tag{B.22}$$

Invoking (B.12) one has that for the points $x_i, x_{s+i} \in I$ there are indices ℓ_i, ℓ_{s+i} for which

$$P_{\ell_i} < x_i \leq P_{\ell_i+1} \quad \text{and} \quad P_{\ell_{s+i}} < x_{s+i} \leq P_{\ell_{s+i}+1} \qquad (1 \leq i \leq n+1). \tag{B.23}$$

Combining (B.22) together with the definition (B.11) of P_ℓ and using the fact that for each index ℓ we have $P_{\ell+1} - P_\ell = P^{1/2}$, one can deduce that

$$\begin{aligned} CP^{1/2} \geq \left|\sum_{i=1}^{n+1} (x_i - x_{s+i})\right| &\geq \left|\sum_{i=1}^{n+1} \left(P_{\ell_i} - P_{\ell_{s+i}}\right)\right| - (n+1)P^{1/2} \\ &\geq \left(\left|\sum_{i=1}^{n+1} (\ell_i - \ell_{s+i})\right| - (n+1)\right) P^{1/2}. \end{aligned}$$

From the above computation we deduce that

$$\left|\sum_{i=1}^{n+1} (\ell_i - \ell_{s+i})\right| \leq C + (n+1) =: C'. \tag{B.24}$$

Return now to the problem of counting the number of solutions of the system (B.21) subject to the restrictions given in (B.22). To do so, we use appropriate generating functions. We write $\mathcal{S} \subset I^{2(n+1)}$ to denote the set of lattice points which satisfy the first two restrictions given in (B.22). By Lemma A.1 with such $\mathcal{S}$ and $I_2 = \widetilde{I}_{\ell_0},\ \boldsymbol{\delta} = \boldsymbol{\delta}_2$ one has

$$V_s^\star\left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2\right) \ll \oint \left|H_{\mathcal{S}}(\boldsymbol{\alpha}) \widetilde{f}_{\ell_0}(\boldsymbol{\alpha})^{2s-2(n+1)}\right| \mathrm{d}\boldsymbol{\alpha}, \tag{B.25}$$

where the exponential sum $H_{\mathcal{S}}(\boldsymbol{\alpha})$ is given by

$$H_{\mathcal{S}}(\boldsymbol{\alpha}) = \sum_{\mathbf{x} \in \mathcal{S}} e\left(\alpha_1(x_1 - x_{s+1}) + \cdots + \alpha_n(x_n^n - x_{s+n}^n) + \alpha_\theta(x_{n+1}^\theta - x_{s+n+1}^\theta)\right),$$

where $\mathbf{x} = (x_1, \ldots, x_{n+1}, x_{s+1}, \ldots, x_{s+n+1})$. One can tile $\mathcal{S}$ by invoking the cover $\left(\widetilde{I}_\ell\right)_\ell$. Taking into account (B.24) we infer that

$$\begin{aligned} |H_{\mathcal{S}}(\boldsymbol{\alpha})| &\ll \sum_{\ell_1=1}^{\lfloor\sqrt{P}\rfloor+1} \cdots \sum_{\substack{\ell_{s+n+1}=1 \\ |\sum_{i=1}^{n+1}(\ell_i - \ell_{s+i})| \leq C'}}^{\lfloor\sqrt{P}\rfloor+1} \prod_{i=1}^{n+1} |\widetilde{f}_{\ell_i}(\boldsymbol{\alpha}) \widetilde{f}_{\ell_{s+i}}(\boldsymbol{\alpha})| \\ &\ll P^{\frac{n+1}{2}} \prod_{i=1}^{n+1} |\widetilde{f}_{\ell_i}(\boldsymbol{\alpha}) \widetilde{f}_{\ell_{s+i}}(\boldsymbol{\alpha})|. \end{aligned}$$

One can bound above the right hand side of (B.25) to obtain

$$V_s^\star\left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2\right) \ll P^{\frac{n+1}{2}} \oint \left(\prod_{i=1}^{n+1} \left|\widetilde{f}_{\ell_i}(\boldsymbol{\alpha})\right| \left|\widetilde{f}_{\ell_{s+i}}(\boldsymbol{\alpha})\right|\right) \left|\widetilde{f}_{\ell_0}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \mathrm{d}\boldsymbol{\alpha}. \tag{B.26}$$

Recall the elementary inequality $|z_1 \cdots z_n| \ll |z_1|^n + \cdots + |z_n|^n$, which is valid for all complex numbers z_i. Using this inequality we obtain that

$$\left(\prod_{i=1}^{n+1} \left|\widetilde{f}_{\ell_i}(\boldsymbol{\alpha})\right| \left|\widetilde{f}_{\ell_{s+i}}(\boldsymbol{\alpha})\right|\right) \left|\widetilde{f}_{\ell_0}(\boldsymbol{\alpha})\right|^{2s-2(n+1)} \ll \left|\widetilde{f}_{\ell_1}(\boldsymbol{\alpha})\right|^{2s} + \cdots + \left|\widetilde{f}_{\ell_{s+n+1}}(\boldsymbol{\alpha})\right|^{2s} + \left|\widetilde{f}_{\ell_0}(\boldsymbol{\alpha})\right|^{2s}.$$

Hence, by (B.26) we infer that

$$V_s^\star\left(I^{(n+1)}, \widetilde{I}_{\ell_0}; \boldsymbol{\delta}_2\right) \ll P^{\frac{n+1}{2}} \oint \left|\widetilde{f}_\ell(\boldsymbol{\alpha})\right|^{2s} \mathrm{d}\boldsymbol{\alpha},$$

where ℓ is one of the indices $\ell_1, \ldots, \ell_{s+n+1}, \ell_0$. Combining the above estimate with (B.20) and invoking (B.17) we deduce that

$$V_s\left(I; \boldsymbol{\delta}_2\right) \ll P^{s-\frac{n+1}{2}} \oint \left|\widetilde{f}_\ell(\boldsymbol{\alpha})\right|^{2s} \mathrm{d}\boldsymbol{\alpha}. \tag{B.27}$$

Appealing once more to Lemma A.1 with $I_1 = I_2 = \widetilde{I}_\ell$ and $\boldsymbol{\delta} = \boldsymbol{\delta}_2$ we see that

$$\oint \left|\widetilde{f}_\ell(\boldsymbol{\alpha})\right|^{2s} \mathrm{d}\boldsymbol{\alpha} \ll V_s\left(\widetilde{I}_\ell; \boldsymbol{\delta}_2\right),$$

where $V_s\left(\widetilde{I}_\ell; \boldsymbol{\delta}_2\right)$ denotes the number of integer solutions of the system

$$\begin{cases} |\sigma_{s,\theta}(\mathbf{x})| < \dfrac{1}{2} & \\ |\sigma_{s,j}(\mathbf{x})| < \dfrac{1}{2} & (1 \le j \le n), \end{cases} \tag{B.28}$$

with $x_i, x_{s+i} \in \widetilde{I}_\ell \ (1 \le i \le s)$. Therefore, the estimate (B.27) now delivers

$$V_s\left(I; \boldsymbol{\delta}_2\right) \ll P^{s-\frac{n+1}{2}} V_s\left(\widetilde{I}_\ell; \boldsymbol{\delta}_2\right). \tag{B.29}$$

We emphasise here, that our choice of $1 \le \ell \le \lfloor\sqrt{P}\rfloor + 1$ is now fixed.

Step4: Taylor series expansion. It is apparent that system (B.28) is equivalent to the system

$$\begin{cases} \left|\displaystyle\sum_{i=1}^{s}\left(x_i^\theta - x_{s+i}^\theta\right)\right| < \dfrac{1}{2} & \\ \displaystyle\sum_{i=1}^{s}\left(x_i^j - x_{s+i}^j\right) = 0 & (1 \le j \le n). \end{cases} \tag{B.30}$$

We substitute $y_i = x_i - Q_\ell \ (1 \le i \le 2s)$, where $Q_\ell = \lfloor P_\ell \rfloor$ and the y_i satisfies the relation $0 < y_i < \lfloor\sqrt{P}\rfloor + 1$. By the Binomial theorem, we see that a tuple **x** satisfies (B.30) if and only if it satisfies the system

$$\begin{cases} \left|\displaystyle\sum_{i=1}^{s}\left((y_i + Q_\ell)^\theta - (y_{s+i} + Q_\ell)^\theta\right)\right| < \dfrac{1}{2} & \\ \displaystyle\sum_{i=1}^{s}\left(y_i^j - y_{s+i}^j\right) = 0 & (1 \le j \le n). \end{cases}$$

One can now apply the argument presented in Theorem 2.3.4 to deal with the inequality. So, we deduce that

$$V_s\left(\widetilde{I}_\ell; \boldsymbol{\delta}_2\right) \ll Z_{s,k,n}\left(Y; \mathbf{h}\right), \tag{B.31}$$

where $Z_{s,k,n}\left(Y;\mathbf{h}\right)$ denotes the number of integer solutions of the system

$$\begin{cases} \left|b_1 Q_\ell^{\theta-1} h_1 + \cdots + b_k Q_\ell^{\theta-k} h_k\right| < 1 \\ \sum_{i=1}^{s} \left(y_i^j - y_{s+i}^j\right) = h_j \quad (1 \le j \le k) \\ \sum_{i=1}^{s} \left(y_i^j - y_{s+i}^j\right) = 0 \quad (1 \le j \le n). \end{cases}$$

with $0 < y_i \le Y$ $(1 \le i \le 2s)$ where $Y = 1 + \lfloor P^{\frac{1}{2}} \rfloor$. Note that the integers h_j satisfy the relation $|h_j| \le sY^j \quad (1 \le j \le k)$.

Step5: Invoking VMVT. From now on we write

$$\mathbf{h} = (h_1, \ldots, h_n, h_{n+1}, \ldots, h_k) = (\mathbf{h}_1, \mathbf{h}_2) \in \mathbb{Z}^n \times \mathbb{Z}^{k-n}.$$

We write $W_{s,k,n}\left(Y;\mathbf{h}\right)$ to denote the number of integer solutions of the system

$$\begin{cases} \sum_{i=1}^{s} \left(y_i^j - y_{s+i}^j\right) = h_j \qquad (1 \le j \le k) \\ \sum_{i=1}^{s} \left(y_i^j - y_{s+i}^j\right) = 0 \qquad (1 \le j \le n), \end{cases} \tag{B.32}$$

with $0 < y_i \le Y$. Moreover, and following the notation of Lemma B.2, we put

$$\mathcal{H}(\mathbf{h}_2) = b_{n+1} Q_\ell^{\theta-(n+1)} h_{n+1} + \cdots + b_k Q_\ell^{\theta-k} h_k.$$

It is apparent that one has $h_j = 0$ for $1 \le j \le n$. So we obtain

$$Z_{s,k,n}\left(Y;\mathbf{h}\right) = \sum_{\substack{|\mathcal{H}(\mathbf{h}_2)|<1 \\ |h_j| \le sY^j \\ n+1 \le j \le k}} W_{s,k,n}\left(Y;\mathbf{h}\right). \tag{B.33}$$

Our aim now is to find an upper bound for the quantity $W_{s,k,n}\left(Y;\mathbf{h}\right)$. We extend the notation $\sigma_{s,j}(\mathbf{x})$ from (B.7) to all indices j with $1 \le j \le k$. As we already mentioned, by the shape of the system (B.32), one has that each h_j $(1 \le j \le n)$ assumes only one value, namely the zero value. Thus, one has

$$\begin{aligned} W_{s,k,n}\left(Y;\mathbf{h}\right) &\ll \sum_{\substack{|h_{n+1}| \le sY^{n+1} \\ h_{n+1} = \sigma_{s,n+1}(\mathbf{x})}} \cdots \sum_{\substack{|h_k| \le sY^k \\ h_k = \sigma_{s,k}(\mathbf{x})}} 1 \\ &\ll Y^{-\frac{1}{2}n(n+1)} \sum_{\substack{|h_j| \le sY^j \\ h_j = \sigma_{s,j}(\mathbf{x}) \\ 1 \le j \le k}} 1. \end{aligned} \tag{B.34}$$

The sum appearing in the right hand side of (B.34) counts the number of integer solutions of the system

$$\sigma_{s,j}(\mathbf{x}) = h_j \qquad (1 \leq j \leq k),$$

with $0 < y_i \leq Y$. We denote this number by $J_{s,k}(Y;\mathbf{h})$. By orthogonality one has

$$J_{s,k}(Y;\mathbf{h}) = \int_{[0,1)^k} \left| \sum_{0<y_i\leq Y} e\left(\alpha_1 y + \cdots + \alpha_k y^k\right)\right|^{2s} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\mathrm{d}\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}\cdot\mathbf{h}$ stands for the standard dot product in $\mathbb{R}^k$. So by the triangle inequality and in view of Theorem 2.3.1 one has for any fixed $\epsilon > 0$ that

$$J_{s,k}(Y;\mathbf{h}) \leq J_{s,k}(Y) \ll Y^{2s-\frac{1}{2}k(k+1)+\epsilon}.$$

Using the above estimate to bound the right hand of (B.34) yields,

$$W_{s,k,n}(Y;\mathbf{h}) \ll Y^{2s-\frac{1}{2}k(k+1)-\frac{1}{2}n(n+1)+\epsilon}. \tag{B.35}$$

Putting together (B.33), (B.35), Lemma B.2 and recalling that $Y = 1 + \lfloor\sqrt{P}\rfloor \ll \sqrt{P}$, we deduce that

$$\begin{aligned} Z_{s,k,n}(Y;\mathbf{h}) &\ll P^{\frac{1}{4}k(k+1)-\frac{1}{4}n(n+1)-\theta+\frac{n+1}{2}} \cdot P^{s-\frac{1}{4}k(k+1)-\frac{1}{4}n(n+1)+\epsilon} \\ &\ll P^{s-\frac{1}{2}n(n+1)-\theta+\frac{n+1}{2}+\epsilon}, \end{aligned}$$

which when incorporated into (B.31) delivers

$$V_s\left(\widetilde{I}_\ell;\boldsymbol{\delta}_2\right) \ll P^{s-\frac{1}{2}n(n+1)-\theta+\frac{n+1}{2}+\epsilon}. \tag{B.36}$$

Finally, combining (B.36) with (B.29) yields

$$V_s(I;\boldsymbol{\delta}_2) \ll P^{2s-\frac{1}{2}n(n+1)-\theta+\epsilon},$$

which in view of (B.9) completes the proof. □

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